Your Pentagonal Mumber Dheorem

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Generating Functions

Andy introduced generating functions through our familiar old friend, the Fibonacci numbers, which have $F(x) = \sum_{k=0}^{\infty} F_k x^k$ as their generating function. To find a closed form for $F(x)$, we use the recurrence relation $F_n = F_{n-1} + F_{n-2}$:

$$\begin{aligned} xF(x) &= \sum_{k=0}^{\infty} F_k x^{k+1} = \sum_{k=1}^{\infty} F_{k-1} x^k = x + \sum_{k=2}^{\infty} F_{k-1} x^k \\ x^2 F(x) &= \sum_{k=0}^{\infty} F_k x^{k+2} = \sum_{k=2}^{\infty} F_{k-2} x^k \\ (x^2 + x)F(x) &= x + \sum_{k=2}^{\infty} (F_{k-2} + F_{k-1}) x^k = x + \sum_{k=2}^{\infty} F_k x^k = \sum_{k=1}^{\infty} F_k x^k \\ &= F(x) - 1 \\ F(x) &= \frac{1}{1 - x - x^2} \end{aligned}$$

Partitions

We learned about some infinite products which are generating functions for the number of partitions with some restrictions:

$$\begin{aligned} H(x) &= \prod_{n=1}^{\infty} (1 + x^n) \\ &= 1 + x + x^2 + x^{1+2} + x^3 + x^{1+3} + x^4 + x^{1+4} + x^{2+3} + x^5 + \cdots \end{aligned}$$

is the generating function for the number of partitions into distinct parts.

$$Q(x) = \sum_{n=1}^{\infty} q_n x^n = \prod_{n=1}^{\infty} (1 - x^n)$$

is the generating function for the number of partitions into an even number of parts − the number of partitions into an odd number of parts.

We also have

$$R(x) = \sum_{n=1}^{\infty} r_n x^n = \prod_{n=1}^{\infty} (1 - ix^n),$$

where $Re(r_n)$=Number of partitions into $4k$ parts − partitions into $4k+2$ parts and $Im(r_n)$=Number of partitions into $4k+1$ parts − partitions into $4k+3$ parts. We can experiment with putting other roots of unity into the infinite product.

Pentagonal Numbers

We found the first few terms in the expansion of $Q(x)$:

$$Q(x) = 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + \cdots$$

and noticed that many terms disappear when we expand $Q(x)$, and the coefficients of other terms are ± 1.

Euler's Pentagonal Number Theorem.

$$\prod_{n=1}^{\infty} (1 - x^n) = 1 + \sum_{k=1}^{\infty} (-1)^k \left(x^{\frac{k(3k+1)}{2}} + x^{\frac{k(3k-1)}{2}}\right)$$

Proof:

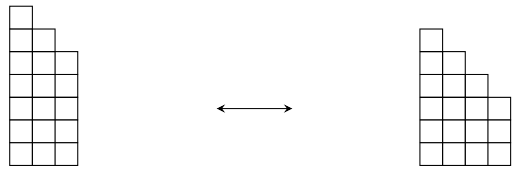


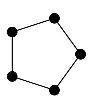
Diagram A Diagram B

We try to construct a bijection between partitions of a given n into an odd number of parts (an 'odd' partition), and partitions of n into an even number of parts (an 'even' partition). We use diagrams where the columns strictly decrease in length from left to right to uniquely represent every partition. In diagram A, we have an odd partition of 18: $7+6+5$. If we shift the top diagonal, starting from the top left block, down to form a new rightmost column, we get diagram B, an even partition of 18: $6+5+4+3$. This has an inverse: we shift the rightmost column in B to form a diagonal on top of the left three rows to get A.

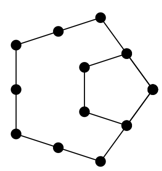
Let d, l be the length of the diagonal and rightmost column, respectively.
If $d < l-1$, then shift the diagonal down to make a new rightmost column.
If $d > l$, then you do the opposite: stack the rightmost column as a diagonal on top of the l leftmost columns.

If the diagram of a partition of n results in $d = l$ or $d = l-1$, we cannot move the diagonal or the row to create a new diagram. Thus, this partition does not correspond to a partition into a different parity of parts. Thus, x^m will appear in $Q(x)$.

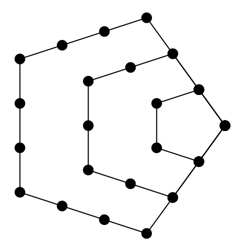
The case $d = l$ will occur when n is a pentagonal number. p_k, the k^{th} pentagonal number, is in the form $\frac{(k)(3k-1)}{2}$. p_1, p_2, p_3 can be represented by the following diagrams:



$p_1 = 5$



$p_2 = 12$



$p_3 = 22$

The case $d = l-1$ will occur when $n = q_j = \frac{(j)(3j+1)}{2}$ for $f \geq 0$. q_j is the j^{th} term in 2, 7, and 15, ..., which are the second pentagonal numbers. The sequence of all p_ks and m_js is known as the generalized pentagonal numbers.

Back to partitions

This theorem implies a recurrence relation for P_n, the number of (not necessarily distinct) partitions of n:

$$P_n = P_{n-1} + P_{n-2} - P_{n-5} - P_{n-7} + \cdots$$

To prove this, notice that:

$$\sum_{k=1}^{\infty} P_k x^k = (1 + x + x^2 + x^3 + \cdots)(1 + x^2 + x^4 + \cdots) \cdots \cdot$$

$$= \prod_{k=1}^{\infty}(1 + x + x^{2k} + \cdots) = \prod_{k=1}^{\infty} \frac{1}{1-x^k} = \frac{1}{Q(x)}$$

More formally, we can write the recurrence relation as

$$P_n = \sum_{m=1}^{\infty} (-1)^{m-1}(P_{n-p_m} + P_{n-q_m}),$$

where q_m is the m^{th} second pentagonal number, and p_m is the m^{th} pentagonal number.