Mini: You Need a Lemma!

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Monday (Yellow Pigs Day!)

We began the week by analysing the similarities between different mathematical objects. Sets, groups, integers, and many more objects can all be represented by dots, with arrows between them describing functions, homomorphisms, or $\leq$, respectively. When we do this, we find that the resulting diagrams are almost identical, regardless of the underlying objects being analyzed. From this observation, we can define **categories**, comprising entire classes of mathematical objects.

A category consists of two types of things: **objects** and **morphisms**. When drawn as a diagram on the board, objects are the dots, and morphisms are the arrows. (Morphisms are similar to functions, but the word "function" is reserved for the category of sets). All categories must obey two properties:

1. Composition: if $f : a \to b$ and $g : b \to c$ then $\exists\ h : a \to c$, such that $gf = h$. Also, composition is associative, so $(fg)(h) = (f)(gh)$ for all $f,\ g,\ h$ morphisms.

2. Identity: $\forall$ objects a, $\exists\ 1_a : a \to a$, such that $1_a f = f$ and $g1_a = g$ for all $f,\ g$ morphisms.

We also define a functor to be a way of mapping categories to other categories; e.g. a functor $F : A \to B$ where A and B are categories. Functors must obey the following rules: $\forall\ a \in A$, $F(a) \in B$, and $\forall\ f : a \to a'$, $F(f) : F(a) \to F(a')$, in addition to preserving the above properties of composition and identity.

Tuesday

The next day, we discussed two specific functors. We first considered the "Free" Functor, which maps the category of sets to the category of groups. Specifically, Free(X) is the free group generated by the elements of X. For example, Free($\{a,\ b\}$) $= \{i,\ a,\ b,\ a^{-1},\ b^{-1},\ ab, ...\}$. Now, if f is a morphism in Set (the category of sets) between X and Y, Free(f) should be a morphism in Group (the category of groups) between Free(X) and Free(Y). In fact, for an element $x \in X$, Free(f)($[x]$) $= [f(x)]$, where $[x]$ is the corresponding group element of x, with x being a set element. (For a more visual representation of this, see Figure 2).

After building our intuition with the Free Functor, we proceeded to a more complex functor: $\mathrm{Mor}(a,\ -) : C \to \mathrm{Set}$, where C is an arbitrary category and a is an object in C. For an object b, $\mathrm{Mor}(a,\ b)$ is defined as the set of all morphisms from a to b. For a morphism $f : b \to c$, $\mathrm{Mor}(a, f)$ is a function from $\mathrm{Mor}(a,\ b)$, the set of all morphisms from a to b, to $\mathrm{Mor}(a,\ c)$, the morphisms from a to c.

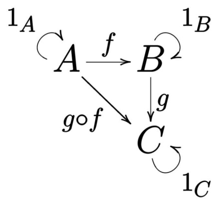

Figure 1: A diagram of a category

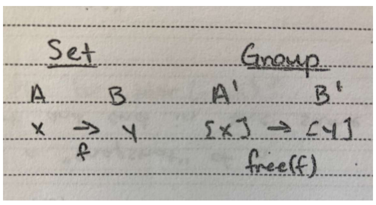

Figure 2: The "Free" Functor from Set to Group

Wednesday

On Wednesday, we continued with the Mor functor. Extending from above, we found that $\mathrm{Mor}(a,\ f) = f \circ -$, which is a function that means "postcompose me with f". For example, $(f \circ -)(g) = f \circ g$. We then considered the very similar $\mathrm{Mor}(-,\ a)$ functor. When the argument is an object, it is defined the same as above; e.g. $\mathrm{Mor}(b,\ a)$ is the set of all morphisms from b to a. On the other hand, $\mathrm{Mor}(f,\ a) = - \circ f$, meaning "precompose me with f". From our experiences with the Mor functors, we came across two new definitions:

Definition 1. *A **covariant** functor F obeys the following relation: if $f : b \to c$, $F(f) : F(b) \to F(c)$.*

Definition 2. *A* ***contravariant*** *functor F obeys the following relation: if $f : b \to c$, $F(f) : F(c) \to F(b)$.*

Using the above definitions, Mor$(a, -)$ is covariant, and Mor$(-, a)$ is contravariant. Additionally, it can be checked that both functors preserve identity and composition, and thus they are valid functors.

We concluded the class by learning about isomorphisms, or isos, which are morphisms with inverses.

Thursday

Day 4 of our Mini was comprised of cones, products, and limits, which are defined as follows:

Definition 3. *A* ***cone*** *over a diagram D is a diagram like Figure 3, where every object in D has a unique mapping from an object, called the* ***apex****. Additionally, all triangles in a cone must commute.*

Definition 4. *The* ***product*** *of x, $y \in C$ is the apex of the "best", or "tightest" cone over the diagram consisting of only x and y, and no morphisms. The product is also called the* ***limit*** *of this diagram. A cone is the "best" if the apex of any other cone can be uniquely mapped to its apex, and the paths of the worse cone can be factored through those of the best cone.*

Using these definitions, we split into three groups and worked on different exercises, finding cones on certain diagrams.

Friday

The next day, we shared our work from Thursday to the class. Using our results, we then discussed a couple more definitions:

Definition 5. *The* ***nadir****, or coapex, of a cone* under *a diagram is a point to which all objects in the diagram can be mapped, such that all triangles commute. The* ***coproduct*** *and* ***colimit*** *are defined identically to the product and limit, but for cones under a diagram (cocones?) instead of over.*

We also learned about **equalizers**, which are the limits of certain diagrams. In Figure 4, for example, E is the equalizer of the diagram, meaning that it consists of all the elements of A that f and g agree on. Finally, we proved that if a diagram has ≥ 2 limits, any two of them are isomorphic.

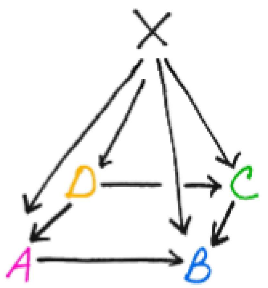

Figure 3: A cone with apex X

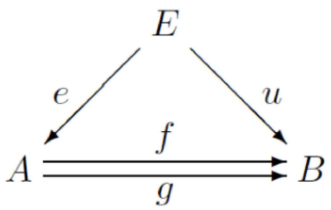

Figure 4: A diagram and its equalizer E

Monday

The penultimate day of first-half Minis was devoted to preparation for the Yoneda (You Need a) Lemma, the holy grail of category theory and the "only non-trivial fact", according to Rhea. We began by learning about natural transformations, which are morphisms between functors. Consider two functors $F, G : C \to D$, where C and D are categories, and two objects $X, Y \in C$, as shown in Figure 5. Also imagine a morphism f in C such that $f : X \to Y$. Then a natural transformation is defined as follows:

Definition 6. *A* ***natural transformation*** *$F \Rightarrow G$ is comprised of component morphisms denoted $\eta_X : F(X) \to G(X)$, such that the diagram in Figure 5 commutes. That is, $\eta_Y(F(f)) = G(f(\eta_X))$.*

We also learned one more definition:

Definition 7. *A functor F is **representable** if it is isomorphic to a Mor functor. We say a **represents** F if:*

- $\exists\, \eta : F \Rightarrow$ *Mor*$(a,\ -)$ *(if F is covariant), OR*
- $\exists\, \eta : F \Rightarrow$ *Mor*$(-,\ a)$, *(if F is contravariant)*

... such that η_b is an isomorphism $\forall\, b \in C$.

We concluded by considering some examples of this, including an excursion into graph theory: an n-coloring of a graph can be thought of as a functor represented by K_n.

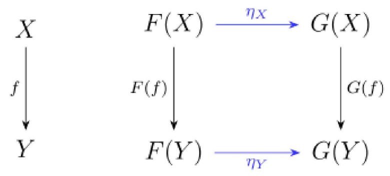

Figure 5: A natural transformation from F to G

Tuesday

At last, we arrived at our ultimate goal: the Yoneda Lemma:

Lemma 1. *For a covariant functor F, $Mor_{Set^C}(Mor_C(a,\ -),\ F) \cong F(a)$, where the category of Mor is denoted by its subscript, and Set^C is the category of functors from C to Set. Similarly, for a contravariant F, $Mor_{Set^C}(Mor_C(-,\ a),\ F) \cong F(a)$.*

The proof of this had many steps. (We also only proved the covariant case). First, we proved that a function ϕ existed from $\text{Mor}_{Set^C}(Mor_C(a,\ -),\ F)$ to $F(a)$. Then we proved that another function ψ existed in the opposite direction. Next, we showed that ψ and ϕ were inverses, so $\psi\phi = 1$ and $\phi\psi = 1$. Finally, we checked that the transformation ψ was natural. We then concluded with a short discussion of the Yoneda Embedding, a functor that converts everything to Mor sets, and whose existence is proven by the Yoneda Lemma.