A new integrable system for a linear Poisson structure and its quantization

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Abstract

In this paper, we consider a Lie subalgebra $\mathfrak{p}$ of the direct sum $\mathfrak{gl}_n \times \mathfrak{gl}_n$ of the general Lie algebra. Let $\mathrm{Sym}(\mathfrak{p})$ be the symmetric algebra equipped with the canonical linear Poisson bracket $\{\cdot,\cdot\}_0$ induced by the Lie bracket on $\mathfrak{p}$. We define a new Liouville integrable system, *i.e.*, a maximally Poisson commutative subalgebra, of the Poisson algebra $(\mathrm{Sym}(\mathfrak{p}, \{\cdot,\cdot\}_0)$. We also study the quantum integrable system, *i.e.*, a maximally commutative subalgebra of the universal enveloping algebra $U(\mathfrak{p})$, corresponding to the new classical integrable system.

Keywords: Liouville integrable system, linear Poisson bracket, log-canonical Poisson brackets, quantum integrable system

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1 Introduction

Let $\mathfrak{gl}_n$ be the general linear Lie algebra. Let $\mathfrak{b}_+$ and $\mathfrak{b}_- \subset \mathfrak{gl}_n$ be the Lie subalgebras consisting of the upper and lower $n \times n$ triangular matrices, respectively. Let

$$\mathfrak{p} := \{(X,Y) \in \mathfrak{b}_- \times \mathfrak{b}_+ \mid \delta(X) = \delta(Y)\},$$

where $\delta(X)$ is the diagonal part of a $n \times n$ matrix X, be the Lie subalgebra of the direct product $\mathfrak{b}_- \times \mathfrak{b}_+$. The symmetric algebra $\mathrm{Sym}(\mathfrak{p})$ is equipped with the Poisson bracket (forming the so-called Lie-Poisson structure) which is determined by the condition that its value $\{X,Y\}$ on the basis elements coincides with the commutator $[X,Y]$ in $\mathfrak{p}$. More precisely, if we take the standard linear basis $\{x_{ij}\}_{1\le i\le j\le n}$ and $\{x_{kl}\}_{1\le l\le k\le n}$ of $\mathfrak{b}_+$ and $\mathfrak{b}_-$ respectively, then we can write $\mathrm{Sym}(\mathfrak{p}) = \mathbb{R}[x_{11},...,x_{nn}]$, and the Poisson bracket on the linear functions are

$$\{x_{ij}, x_{kl}\}_0 = \delta_{jk}x_{il} - \delta_{il}x_{kj}, \text{ for } 1 \le i \le j \le n,\ 1 \le k \le l \le n, \tag{1}$$
$$\{x_{ij}, x_{kl}\}_0 = \delta_{jk}x_{il} - \delta_{il}x_{kj}, \text{ for } 1 \le j \le i \le n,\ 1 \le l \le k \le n, \tag{2}$$
$$\{x_{ij}, x_{kl}\}_0 = 0, \text{ for } 1 \le i < j \le n,\ 1 \le l < k \le n. \tag{3}$$

Remark 1.1. If we view the basis element x_{ij} as a linear functional on the dual space $\mathfrak{p}^*$, then the bracket we just introduced is the so-called linear Poisson structure for the vector space $\mathfrak{p}^*$. The meaning of the index "0" in $\{\cdot,\cdot\}_0$ will be clarified later.

1.1 A new classical integrable system of the Poisson algebra $(\mathrm{Sym}(\mathfrak{p}), \{\cdot,\cdot\}_0)$

We refer, by an integrable system of the Poisson algebra $(\mathrm{Sym}(\mathfrak{p}), \{\cdot,\cdot\}_0)$, to a maximally commutative Poisson subalgebra. This will be recalled more formally in Definition 4.1, where we define the notion of a Liouville integrable system of $(\mathrm{Sym}(\mathfrak{p}), \{\cdot,\cdot\}_0)$. To construct such a subalgebra, it suffices to construct its generators.

Let $X = (x_{kl})_{k,l=1,...,n}$ be the $n \times n$ matrix collecting the generators of $\mathrm{Sym}(\mathfrak{p})$. For any integers $1 \le i,j \le n$, let $\Delta_{[1,i],[j,...,j+i-1]}$ denote the $i \times i$ minor of the matrix X formed by the first i rows and columns $j,...,j+i-1$. Notice that $\Delta_{[1,i],[j,...,j+i-1]}$ is a degree i polynomial in the variables x_{kl}, which is an element in $\mathrm{Sym}(\mathfrak{p})$. Our first main theorem states

Theorem 1.2 (Theorem 4.4)**.** *The $\frac{n(n+1)}{2}$ functions in the set*

$$\mathbf{D} = \left\{\Delta_{[k-i+1,k],[1,i]} : 1 \le i \le \lfloor \frac{n}{2} \rfloor, 2i \le k \le n\right\} \cup \left\{\Delta_{[1,i],[k-i+1,k]} : 1 \le i \le \lfloor \frac{n}{2} \rfloor, 2i \le k \le n\right\},$$

and the set

$$\mathbf{C} = \{\mathrm{tr}(X)\} \cup \left\{ \sum_{k\in[i+1,n-i]} (\Delta_{[n-i+1,n],[1,i]}\Delta_{k\cup[1,i],k\cup[n-i+1,n]} + \Delta_{[1,i],[n-i+1,n]}\Delta_{k\cup[n-i+1,n],k\cup[1,i]}) : 1 \le i \le \lfloor \frac{n-1}{2} \rfloor \right\}$$

form a Liouville integrable system of $(\mathrm{Sym}(\mathfrak{p}), \{\cdot,\cdot\}_0)$.

Let us explain where the functions come from, and outline our proof of the above theorem.

Firstly, there is another Poisson bracket, denoted by $\{\cdot,\cdot\}$ (without the lower index 0), on the symmetric algebra $\mathrm{Sym}(\mathfrak{p}) = \mathbb{R}[x_{11},...,x_{nn}]$. Such a Poisson algebra has been extensively studied in the literature, in relation to Poisson-Lie groups [6], integrable systems [7], and so on. In terms of $\{x_{ij}\}$, the bracket takes a polynomial form, *i.e.*, there exists polynomials $P_{ij,kl} \in \mathbb{R}[x_{11},...,x_{nn}]$ for all $i,j,k,l = 1,...,n$ such that

$$\{x_{ij}, x_{kl}\} = P_{ij,kl}(x_{11},...,x_{nn}). \tag{4}$$

The explicit form shall be given in Section 2.1. An important aspect of this Poisson bracket is that it has a log-canonical structure, *i.e.*, there exists a set of algebraically independent elements $M_1, ..., M_{n^2}$ in the symmetric algebra $\mathrm{Sym}(\mathfrak{p})$, such that the bracket $\{\cdot,\cdot\}$ takes the simple form

$$\{M_i, M_j\} = \pi_{ij} M_i M_j, \quad \forall 1 \leq i, j \leq n^2, \tag{5}$$

where π_{ij}'s are certain real constants.

It is also known that the $P_{ij,kl}$'s are polynomials without a constant term. Thus, the bracket defined by taking the linear part of the polynomial Poisson bracket $\{\cdot,\cdot\}$ is also a Poisson structure, which will recover the Poisson bracket $\{\cdot,\cdot\}_0$ we previously introduced. Refer to Chapter 7 of [4] and section 2.3 for more detail. More precisely,

$$\{x_{ij}, x_{kl}\}_0 = \text{The linear terms of the polynomial } P_{ij,kl} = \{x_{ij}, x_{kl}\}.$$

Given such a relation, one may consider the following question: Take $\{M_1^{\mathrm{low}}, ..., M_{n^2}^{\mathrm{low}}\}$ to be the set of functions given by the respective lowest degree terms of the polynomials $M_1, ..., M_{n^2}$. Then, what is the analogue of the log-canonical property of $M_1, ..., M_{n^2}$ in the linear Poisson bracket $\{\cdot,\cdot\}_0$? That is, do the identities in (5) impose any conditions on the linear Poisson bracket $\{\cdot,\cdot\}_0$ between $M_1^{\mathrm{low}}, ..., M_{n^2}^{\mathrm{low}}$?

Indeed, in Section 3.1 and Section 4.4.1, we will show that

(1) the lowest degree terms, denoted as $M_1^{\mathrm{low}}, ..., M_{n^2}^{\mathrm{low}}$, of the polynomials $M_1, ..., M_{n^2}$, Poisson commute with each other under the linear Poisson bracket $\{\cdot,\cdot\}_0$, *i.e.*,

$$\{M_i^{\mathrm{low}}, M_j^{\mathrm{low}}\}_0 = 0, \ \forall \ i, j = 1, ..., n^2;$$

(2) the set of nonconstant distinct functions in $\{M_1^{\mathrm{low}}, ..., M_{n^2}^{\mathrm{low}}\}$ coincide with the functions in set $\mathbf{D}$ given in Theorem 1.2.

It is well-known that an integrable system (or equivalently, a maximal Poisson commutative subalgebra) of the Poisson algebra $(\mathrm{Sym}(\mathfrak{p}), \{\cdot,\cdot\}_0)$ is generated by $r + d$ algebraically independent functions, where r is the rank of the Poisson bracket $\{\cdot,\cdot\}_0$ (see Definition 2.10) and d is the cardinality of a complete set of Casimir functions (see Definition 3.4). Since the rank of the Poisson bracket $\{\cdot,\cdot\}_0$ is $\frac{n(n-1)}{2}$ (see Theorem 2.9) and the cardinality of a complete set of Casimir functions is n, by the above results (1) and (2), one needs to construct more functions to get an integrable system; our set $\mathbf{C}$ consists of these functions. The motivation for the construction of $\mathbf{C}$ comes from observations made when we take the lowest degree terms of a complete set of Casimirs. This is explained in more detail in parts 4.3 and 4.4.2.

1.2 A quantum integrable system of $U(\mathfrak{p})$

The universal enveloping algebra $U(\mathfrak{p})$ is an unital associative algebra generated by $\{e_{ij}\}_{i,j=1,...,n}$ with the relations

$$[e_{ij}, e_{kl}] = \delta_{jk} e_{il} - \delta_{il} e_{kj}, \text{ for } 1 \leq i \leq j \leq n, \ 1 \leq k \leq l \leq n, \tag{6}$$

$$[e_{ij}, e_{kl}] = \delta_{jk} e_{il} - \delta_{il} e_{kj}, \text{ for } 1 \leq j \leq i \leq n, \ 1 \leq l \leq k \leq n, \tag{7}$$

$$[e_{ij}, e_{kl}] = 0, \text{ for } 1 \leq i < j \leq n, \ 1 \leq l < k \leq n. \tag{8}$$

The algebra has a canonical filtration

$$U_0(\mathfrak{p}) \subset U_1(\mathfrak{p}) \subset \cdots,$$

where $U_0(\mathfrak{p}) = \mathbb{R}$, and $U_n(\mathfrak{p})$ for $n \in \mathbb{N}$ is the subspace of $U(\mathfrak{p})$ generated by the products of the form $x_1 \cdots x_m$, where the integer $m \leq n$ and $x_i \in \mathfrak{p}$ for all $i \leq n$. The graded algebra $\mathrm{gr}(U(\mathfrak{p}))$ associated to this filtration is commutative and is generated by the image under the natural homomorphism $\iota : \mathfrak{p} \to \mathrm{gr}(U(\mathfrak{p}))$. The map extends to an isomorphism of the symmetric algebra $\mathrm{Sym}(\mathfrak{p})$ to $\mathrm{gr}(U(\mathfrak{p}))$. In the rest of the paper, we will identity $\mathrm{Sym}(\mathfrak{p})$ with $\mathrm{gr}(U(\mathfrak{p}))$ through this isomorphism, and we will denote by $\mathrm{gr}(X)$ the image of $X \in U(\mathfrak{p})$ under the identification $\mathrm{gr}(U(\mathfrak{p})) \cong \mathrm{Sym}(\mathfrak{p})$. For example, we have

$$\mathrm{gr}(e_{12}e_{23} + e_{34}) = x_{12}x_{23}, \quad \mathrm{gr}(e_{12}e_{24} + e_{24}e_{12}) = \mathrm{gr}(2e_{12}e_{24} - e_{14}) = 2x_{12}x_{24} \in \mathrm{Sym}(\mathfrak{p}).$$

Now, let us introduce the quantization problem. Given that the minor functions Δ of $\mathrm{Sym}(\mathfrak{p})$ from the set $\mathbf{D} \cup \mathbf{C}$ are Poisson commutative, the quantization problem is to look for the elements $q\Delta$ of $U(\mathfrak{p})$ which quantize the Δ in the sense that $\mathrm{gr}(q\Delta) = \Delta$, and such that any two elements $q\Delta_1$ and $q\Delta_2$ commute with each other in $U(\mathfrak{p})$.

Let E be an $n \times n$ matrix with entries valued in the universal enveloping algebra $U(\mathfrak{p})$, where

$$(E)_{ij} = e_{ij}, \qquad \text{for } 1 \le i, j \le n, \quad 1 \le m \le k.$$

We would like to define the analogue of the classical minor $\Delta_{[1,i],[j,...,j+i-1]}$ for the matrix E. However, note that the entries of E do not commute with each other, so the determinant of a submatrix of E is not well defined. For example, the row expansion and column expansion determinants of the 2×2 matrix $E_2 = \begin{pmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{pmatrix}$ are $e_{11}e_{22} - e_{12}e_{21}$ and $e_{11}e_{22} - e_{21}e_{12}$ respectively. Since $e_{21}e_{12} - e_{12}e_{21} = e_{22} \neq 0$, the row expansion and column expansion determinants are not equal.

Thus, at the first glimpse, there is seemingly no natural lift of the classical minors to elements in $U(\mathfrak{p})$. However, upon closer inspection, one observes that for $1 \le i \le \lfloor \frac{n}{2} \rfloor, 2i \le k \le n$, all the elements of the submatrix of E formed by the first i rows and $k-i+1, ..., k$ columns commute with each other. Therefore, the minor of E formed by the first i rows and $k-i+1, ..., k$ columns can be defined in the same way as the classical case! For example, for $i = 2$ and $k = 4$, as the elements $e_{13}, e_{14}, e_{23}, e_{24}$ commute, their corresponding 2×2 minor is $e_{13}e_{24} - e_{23}e_{14}$.

We are hence motivated to define the quantum minors as follows: for $1 \le i \le \lfloor \frac{n}{2} \rfloor, 2i \le k \le n$,

$$q\Delta_{[1,i],[k-i+1,k]}(E) := \sum_{\sigma} (-1)^{|\sigma|} e_{1\sigma(k-i+1)} e_{2\sigma(k-i+2)} \cdots e_{i\sigma(k)} \in U(\mathfrak{p}), \tag{9}$$

$$q\Delta_{[k-i+1,k],[1,i]}(E) := \sum_{\sigma} (-1)^{|\sigma|} e_{\sigma(k-i+1)1} e_{\sigma(k-i+2)2} \cdots e_{\sigma(k)i} \in U(\mathfrak{p}), \tag{10}$$

where the summation is over all the permutations σ of $\{k-i+1, k-i, ..., k\}$ and $|\sigma|$ is the signature of a permutation. We remark that all elements $\{e_{ab}\}_{a=1,...,i, b=k-i+1,...,k}$ appearing in (9) commute with each other. Thus, we have the Laplace expansion (see Lemma 5.1.1) for the quantum minors that will be used in the proof of our second main theorem:

Theorem 1.3 (Theorem 5.2)**.** *The quantum minors in the set*

$$q\mathbf{D} = \left\{ q\Delta_{[k-i+1,k],[1,i]}(E) : 1 \le i \le \lfloor \frac{n}{2} \rfloor, 2i \le k \le n \right\} \cup \left\{ q\Delta_{[1,i],[k-i+1,k]}(E) : 1 \le i \le \lfloor \frac{n}{2} \rfloor, 2i \le k \le n \right\},$$

commute with each other. That is,

$$q\Delta_1 \cdot q\Delta_2 = q\Delta_2 \cdot q\Delta_1 \in U(\mathfrak{p}), \ \textit{for all quantum minors } q\Delta_1, q\Delta_2 \in q\mathbf{D}.$$

Note that $q\Delta_{[1,i],[k-i+1,k]}(E)$ is an element of $U(\mathfrak{p})$ with filtered degree i, and its associated graded element coincides with $\Delta_{[1,i],[k-i+1,k]}$. That is,

$$\mathrm{gr}\left(q\Delta_{[1,i],[k-i+1,k]}(E)\right) = \Delta_{[1,i],[k-i+1,k]}(X) \in \mathrm{Sym}(\mathfrak{p}).$$

Therefore, by the above theorem, we have successfully solved the quantization problem for the minors in $\mathbf{D}$. We can also conclude that the subalgebra of $U(\mathfrak{p})$ generated by the elements in $q\mathbf{D}$ is a commutative subalgebra. However, it is not a maximal commutative subalgebra: we still need to find a set $q\mathbf{C}$ of $\lfloor \frac{n+1}{2} \rfloor$ elements in $U(\mathfrak{p})$ such that all elements from $q\mathbf{D} \cup q\mathbf{C}$ commute with each other, and the image of $q\mathbf{C}$ under the isomorphism $\mathrm{gr}(U(\mathfrak{p})) \cong \mathrm{Sym}(\mathfrak{p})$ coincide with $\mathbf{C}$. We have a candidate for the set $q\mathbf{C}$ but did not have enough time to verify it, so we will leave it for a future study.

Our paper is organised as follows: In Section 2, we recall the elementary definitions and fix the notation. In Section 3, we build the key tool for our main theorems. In Section 4, we construct a new integrable system for a linear Poisson structure, which is the major part of our paper. In Section 5, we consider the quantization of our integrable system. In Section 6, we discuss our future work and provide more examples of classical integrable systems constructed by a similar strategy.

2 Preliminaries

In this section, we shall recall some elementary definitions and classical results, which are given in detail in [3], [4], and [6], as well as [1] and [2].

2.1 The standard Poisson bracket on $\mathrm{Sym}(\mathfrak{p})$

In this part, we introduce the standard Poisson structure on the space GL_n of invertible $n \times n$ real matrices. We mainly follow the conventions in [4] and the classical results as summarised by Kosmann-Schwarzbach [3].

Definition 2.1. A *Poisson algebra* is a commutative algebra A, together with a bilinear map $\{\cdot,\cdot\}\colon A \times A \to A$, called a Poisson bracket, satisfying that for all $f, g, h \in A$,

$$\text{(Lebniz identity)} \quad \{fg, h\} = f\{g, h\} + g\{f, h\}, \tag{11}$$
$$\text{(Jacobi identity)} \quad \{f, \{g, h\}\} + \{h, \{f, g\}\} + \{g, \{h, f\}\} = 0. \tag{12}$$

In the rest of the paper, we will write an element $A \in \mathrm{GL}_n$ as $A = \mathrm{Id}_n + X$, where Id_n is the $n \times n$ identity matrix, and $X = (x_{ij})$. In this way, $\{x_{ij}\}_{i,j=1,\dots,n}$ are seen as coordinates on GL_n. Here, each x_{ij} is considered as the function that associates the coefficient of $A - \mathrm{Id}_n$ in the i-th column and j-th row to a matrix $A \in \mathrm{GL}_n$. We will focus on the polynomial functions on Mat_n with generators $\{x_{ij}\}$, *i.e.*,

$$\mathcal{R}_n := \mathbb{R}[x_{11}, \dots, x_{nn}].$$

To define a Poisson bracket on the commutative algebra $\mathcal{R}_n$, it suffices to define the Poisson brackets between the generators x_{ij}'s. This is because by the Leibniz identity (11), for elements f and g in $\mathcal{R}_n$, their Poisson bracket is determined by

$$\{f, g\} = \sum_{a,b,c,d=1}^{n} \{x_{ab}, x_{cd}\} \frac{\partial f}{\partial x_{ab}} \frac{\partial g}{\partial x_{cd}}. \tag{13}$$

We now introduce the so-called *standard* Poisson bracket on $\mathcal{R}_n$. It suffices to know the pairwise Poisson brackets $\{x_{ij}, x_{kl}\}$, and these brackets can be arranged in a $n^2 \times n^2$ matrix, which we denote by $\{X \overset{\otimes}{,} X\}$. By definition,

$$\{X \overset{\otimes}{,} X\} = \begin{pmatrix} \{x_{11}, x_{11}\} & \dots & \{x_{11}, x_{1n}\} & \{x_{12}, x_{11}\} & \dots & \{x_{1n}, x_{1n}\} \\ \{x_{11}, x_{21}\} & \dots & \{x_{11}, x_{2n}\} & \{x_{12}, x_{21}\} & \dots & \{x_{1n}, x_{2n}\} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \{x_{21}, x_{11}\} & \dots & \{x_{21}, x_{1n}\} & \{x_{22}, x_{11}\} & \dots & \{x_{2n}, x_{1n}\} \\ & & & & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \{x_{21}, x_{n1}\} & \dots & \{x_{21}, x_{nn}\} & \dots & \dots & \dots \\ \{x_{n1}, x_{n1}\} & \dots & \{x_{n1}, x_{nn}\} & \dots & \dots & \{x_{nn}, x_{nn}\} \end{pmatrix}.$$

Theorem 2.2 (See *e.g.*, [3])**.** *The brackets, defined on the generators x_{ij}'s by*

$$\{X \overset{\otimes}{,} X\} = (X + \mathrm{Id}_n) \otimes (X + \mathrm{Id}_n) \cdot r - r \cdot (X + \mathrm{Id}_n) \otimes (X + \mathrm{Id}_n),$$

induces a Poisson bracket on the commutative algebra $\mathcal{R}_n$. Here,

$$r = \sum_{1 \leq i < j \leq n} (E_{ij} \otimes E_{ji} - E_{ji} \otimes E_{ij}) \in \mathrm{Mat}_n \otimes \mathrm{Mat}_n,$$

where E_{ij} is the $n \times n$ matrix with all nonzero entries except for a 1 at the $(i, j)^{th}$ position.

We shall denote by $(\mathcal{R}_n, \{\cdot, \cdot\})$ the Poisson algebra we just introduced. Denote the $n^2 \times n^2$ matrix $(X + \mathrm{Id}_n) \otimes (X + \mathrm{Id}_n) \cdot r - r \cdot (X + \mathrm{Id}_n) \otimes (X + \mathrm{Id}_n)$ as $P = (x_{ab,cd})$. As P is obtained from the Kronecker product of matrices, we denote by $P_{ab,cd}$ its entry in the $(a(n-1)+b)^{th}$ row, $(c(n-1)+d)^{th}$ column. Note that $P_{ab,cd}$ is a degree two polynomial in $\mathcal{R}_n$.

Example 2.3. (The standard Poisson brackets for $(\mathcal{R}_3, \{\cdot, \cdot\})$.) In this case, part of the matrix P is

$$\begin{pmatrix} 0 & -x_{11}x_{12} - x_{12} & -x_{11}x_{13} - x_{13} & x_{11}x_{12} + x_{12} & 0 & -x_{12}x_{13} & x_{11}x_{13} + x_{13} & \cdots \\ -x_{11}x_{21} - x_{21} & -2x_{12}x_{21} & -2x_{13}x_{21} & 0 & -x_{12}x_{22} - x_{12} & -2x_{13}x_{22} - 2x_{13} & 0 & \cdots \\ -x_{11}x_{31} - x_{31} & -2x_{12}x_{31} & -2x_{13}x_{31} & 0 & -x_{12}x_{32} & -2x_{13}x_{32} & 0 & \cdots \\ x_{11}x_{21} + x_{21} & 0 & 0 & 2x_{12}x_{21} & x_{12}x_{22} + x_{12} & 0 & 2x_{13}x_{21} & \cdots \\ 0 & -x_{21}x_{22} - x_{21} & -x_{21}x_{23} & x_{21}x_{22} + x_{21} & 0 & -x_{22}x_{23} - x_{23} & x_{21}x_{23} & \cdots \\ -x_{21}x_{31} & -2x_{22}x_{31} - 2x_{31} & -2x_{23}x_{31} & 0 & -x_{22}x_{32} & -2x_{23}x_{32} & 0 & \cdots \\ x_{11}x_{31} + x_{31} & 0 & 0 & 2x_{12}x_{31} & x_{12}x_{32} & 0 & 2x_{13}x_{31} & \cdots \\ x_{21}x_{31} & 0 & 0 & 2x_{22}x_{31} + 2x_{31} & x_{22}x_{32} + x_{32} & 0 & 2x_{23}x_{31} & \cdots \\ 0 & -x_{31}x_{32} & -x_{31}x_{33} - x_{31} & x_{31}x_{32} & 0 & -x_{32}x_{33} - x_{32} & x_{31}x_{33} + x_{31} & \cdots \end{pmatrix}.$$

Some Poisson brackets that we obtain from P are

$$\begin{aligned} &\{x_{11}, x_{12}\} = -x_{11}x_{12} - x_{12}, && \{x_{11}, x_{13}\} = -x_{11}x_{13} - x_{13}, && \{x_{11}, x_{21}\} = x_{11}x_{12} + x_{12}, \\ &\{x_{12}, x_{21}\} = 0, && \{x_{21}, x_{31}\} = 2x_{13}x_{21}, && \{x_{31}, x_{33}\} = x_{13}x_{33} + x_{13}. \end{aligned}$$

2.2 A set of log-canonical elements in $(\mathcal{R}_n, \{\cdot, \cdot\})$

One of the main results of Kogan and Zelevinsky [2] and Gekhtman, Shapiro, and Vainshein [1] showed the generalised construction of systems of log-canonical functions for the Poisson algebra $(\mathcal{R}_n, \{\cdot, \cdot\})$ using certain minors of GL_n. We will hereby introduce such a specific set of log-canonical minors, which will be used in our discussion in this paper.

Definition 2.4. Consider the Poisson algebra $(\mathcal{R}_n, \{\cdot, \cdot\})$. A set of algebraically independent functions $\{g_1, \ldots, g_{n^2}\}$ in $\mathcal{R}_n$ is *log-canonical* with respect to the Poisson structure $\{\cdot, \cdot\}$ if

$$\{g_i, g_j\} = \pi_{ij} g_i g_j, \quad \forall 1 \leq i, j \leq n^2,$$

where π_{ij}'s are certain real constants.

We will take our specific set of minors of $A = \mathrm{Id}_n + X$, denoted by $M_1, \ldots, M_{n^2}$, to be the functions in the following sets:

1. $\{\Delta_{[1,i],[1,i]}(\mathrm{Id}_n + X) : 1 \leq i \leq n\}$;
2. $\{\Delta_{[k-i+1,k],[1,i]}(\mathrm{Id}_n + X) : 1 \leq i < k \leq n\}$;
3. $\{\Delta_{[1,i],[k-i+1,k]}(\mathrm{Id}_n + X) : 1 \leq i < k \leq n\}$;

where $\Delta_{I,J}(\mathrm{Id}_n + X)$ denotes the minor of the $n \times n$ matrix $\mathrm{Id}_n + X$ with row set I and column set J, and $[a, b]$ denotes the set $\{a, a+1, \ldots, b\}$. For example, $\Delta_{[12],[23]}(\mathrm{Id}_n + X) = x_{12}x_{23} - x_{21}x_{13}$ and $\Delta_{[12],[12]}(\mathrm{Id}_n + X) = (x_{11} + 1)(x_{22} + 1) - x_{21}x_{12}$.

From Theorem 2.6 in Kogan and Zelevinsky [2] and Theorem 3.1 in Gekhtman, Shapiro, and Vainshtein [1], it follows that

Theorem 2.5. *The set of minors* $\{M_1, \ldots, M_{n^2}\}$ *is log-canonical with respect to the Poisson algebra* $(\mathcal{R}_n, \{\cdot,\cdot\})$.

Example 2.6. (Continuation of Example 2.3.) We hereby list the minors of $\mathcal{R}_3$, writing $\Delta_{I,J}(\mathrm{Id}_n + X)$ as $\Delta_{I,J}$ for conciseness:

$$\Delta_{[1],[1]}, \Delta_{[1,2],[1,2]}, \Delta_{[1,3],[1,3]}, \Delta_{[2],[1]}, \Delta_{[3],[1]}, \Delta_{[2,3],[1,2]}, \Delta_{[1],[2]}, \Delta_{[1],[3]}, \Delta_{[1,2],[2,3]}.$$

To illustrate their log-canonical nature, we show the explicit computation for the Poisson bracket between some of our minors:

$$\{\Delta_{[1],[1]}, \Delta_{[1,2],[1,2]}\} = 0, \qquad \{\Delta_{[1],[1]}, \Delta_{[2,3],[1,2]}\} = -\Delta_{[1],[1]}\Delta_{[2,3],[1,2]},$$
$$\{\Delta_{[3],[1]}, \Delta_{[1,2],[1,2]}\} = \Delta_{[3],[1]}\Delta_{[1,2],[1,2]}, \qquad \{\Delta_{[1,2],[1,2]}, \Delta_{[1,2],[2,3]}\} = -\Delta_{[1,2],[1,2]}\Delta_{[1,2],[2,3]}.$$

2.3 The linearization of $(\mathcal{R}_n, \{\cdot,\cdot\})$ is $(\mathrm{Sym}(\mathfrak{p}), \{\cdot,\cdot\}_0)$

In this part, we introduce the key properties of a linear Poisson bracket, which is the Poisson structure that our integrable system is built on. We will first recall the following standard result regarding the linearization of a polynomial Poisson structure given in [4].

Theorem 2.7. *Consider a polynomial Poisson bracket* $\{\cdot,\cdot\}$ *for the polynomial algebra* $\mathbb{R}[x_1, \ldots, x_d]$*. Suppose that the evaluation of the polynomial* $\{x_i, x_j\}$ *at* $x_1 = 0, ..., x_d = 0$ *is zero. Then the bracket*

$$\{x_i, x_j\}_0 := \textit{linear part of the polynomial } \{x_i, x_j\} \tag{14}$$

is a Poisson structure on $\mathbb{R}[x_1, \ldots, x_d]$*. We say that* $\{\cdot,\cdot\}_0$ *is the linearization of* $\{\cdot,\cdot\}$.

It is known (see, *e.g.*, [6]) that the standard Poisson bracket $\{\cdot,\cdot\}$ on $\mathcal{R}_n$ vanishes at $X = 0$. That is, the standard Poisson bracket $\{\cdot,\cdot\}$ on $\mathcal{R}_n = \mathbb{R}[x_{11}, x_{12} \ldots, x_{nn}]$ vanishes at $x_{11} = 0, x_{12} = 0, \ldots, x_{nn} = 0$. Thus, by the previous theorem, $\{\cdot,\cdot\}_0$ is a linear Poisson structure on the commutative algebra $\mathcal{R}_n$.

Theorem 2.8 (See, *e.g.*, [6])**.** *The induced Poisson bracket* $\{\cdot,\cdot\}_0$ *on* $\mathbb{R}[x_{11}, ..., x_{nn}]$ *takes the following form: for the linear functions* x_{ij}, x_{kl},

$$\{x_{ij}, x_{kl}\}_0 = \delta_{jk}x_{il} - \delta_{il}x_{kj}, \ \textit{for } 1 \le i \le j \le n, \ 1 \le k \le l \le n, \tag{15}$$
$$\{x_{ij}, x_{kl}\}_0 = \delta_{jk}x_{il} - \delta_{il}x_{kj}, \ \textit{for } 1 \le j \le i \le n, \ 1 \le l \le k \le n, \tag{16}$$
$$\{x_{ij}, x_{kl}\}_0 = 0, \ \textit{for } 1 \le i < j \le n, \ 1 \le l < k \le n. \tag{17}$$

Note that the above formulae for the bracket $\{\cdot,\cdot\}_0$ coincide with the Poisson brackets given in (1)-(3). Since we have identified the symmetric algebra $\mathrm{Sym}(\mathfrak{p})$ with $\mathcal{R}_n = \mathbb{R}[x_{11}, ..., x_{nn}]$ as stated in the introduction, the Poisson algebra $(\mathrm{Sym}(\mathfrak{p}) = \mathcal{R}_n, \{\cdot,\cdot\}_0)$ is interpreted as the linearization of the Poisson algebra $(\mathcal{R}_n, \{\cdot,\cdot\})$ equipped with the standard Poisson bracket.

Now, let us consider the $n^2 \times n^2$ matrices $P = \{X \overset{\otimes}{,} X\} = (\{x_{ab}, x_{cd}\})$ and $P_0 = (\{x_{ab}, x_{cd}\}_0)$.

Theorem 2.9. *(Chapter 7 of [4].) We have that* $\mathrm{rank}(P) = \mathrm{rank}(P_0) = n^2 - n$*, where* rank *is the generic rank of a polynomial matrix.*

Definition 2.10. The rank of Poisson brackets $\{\cdot,\cdot\}_0$ and $\{\cdot,\cdot\}$ are the rank of the $n^2 \times n^2$ matrices P_0 and P, respectively.

Example 2.11. (Continuation of Example 2.3.) The matrix P_0 is

$$\begin{pmatrix} 0 & -x_{12} & -x_{13} & x_{12} & 0 & 0 & x_{13} & 0 & 0 \\ -x_{21} & 0 & 0 & 0 & -x_{12} & -2x_{13} & 0 & 0 & 0 \\ -x_{31} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -x_{13} \\ x_{21} & 0 & 0 & 0 & x_{12} & 0 & 0 & 2x_{13} & 0 \\ 0 & -x_{21} & 0 & x_{21} & 0 & -x_{23} & 0 & x_{23} & 0 \\ 0 & -2x_{31} & 0 & 0 & -x_{32} & 0 & 0 & 0 & -x_{23} \\ x_{31} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_{13} \\ 0 & 0 & 0 & 2x_{31} & x_{32} & 0 & 0 & 0 & x_{23} \\ 0 & 0 & -x_{31} & 0 & 0 & -x_{32} & x_{31} & x_{32} & 0 \end{pmatrix}.$$

We see that the matrix P_0 indeed takes the linear part of the matrix P of polynomial functions in Example 2.3. Some linear Poisson brackets that we have are

$$\begin{aligned} &\{x_{11}, x_{12}\}_0 = -x_{12}, \quad &&\{x_{11}, x_{13}\}_0 = -x_{13}, \quad &&\{x_{11}, x_{21}\}_0 = x_{12}, \\ &\{x_{12}, x_{21}\}_0 = 0, &&\{x_{21}, x_{31}\}_0 = 0, &&\{x_{31}, x_{33}\}_0 = x_{13}. \end{aligned}$$

3 Constructing Poisson commutative functions for $(\mathcal{R}_n, \{\cdot,\cdot\}_0)$

As proposed by [5], we can construct Poisson commutative functions by taking the lowest degree terms of certain functions. More rigorously, we introduce the following definition given in [5]:

Definition 3.1 (Given in [5]). For a rational function f, f^{low} is given by

(1) The term(s) of the lowest degree in f, when f is a polynomial function;

(2) $\frac{p^{\text{low}}}{q^{\text{low}}}$, when $f = \frac{p}{q}$ for some polynomial functions p and q.

Noting that $(hg)^{\text{low}} = h^{\text{low}} g^{\text{low}}$ for all polynomials h and g, it is easy to check that f^{low} is well-defined when $f = \frac{p}{q}$.

3.1 Poisson commutative functions arising from minors

Recall that we have a set of minors $M_1, \ldots, M_{n^2}$, which are log-canonical in $(\mathcal{R}_n, \{\cdot,\cdot\})$. A key motivation for defining f^{low} is evident in the following observation given by [5].

Proposition 3.2 (Given in [5]). *Suppose that g_1 and g_2 are log-canonical in $(\mathcal{R}_n, \{\cdot,\cdot\})$, then*

$$\{g_1^{low}, g_2^{low}\}_0 = 0.$$

Recall again that $\{\cdot,\cdot\}_0$ is the linearazation of Poisson bracket $\{\cdot,\cdot\}$.

Proof. We will assume for the sake of contradiction that $\{g_1^{\text{low}}, g_2^{\text{low}}\}_0 \neq 0$.

First, we show the easy fact that for any two polynomials f and g, either $\{f, g\} = 0$ or

$$\deg(\{f, g\}) \geq \deg(f^{\text{low}}) + \deg(g^{\text{low}}) - 1. \tag{18}$$

Recall that

$$\{f, g\} = \sum_{a,b,c,d=1}^{n} \{x_{ab}, x_{cd}\} \frac{\partial f}{\partial x_{ab}} \frac{\partial g}{\partial x_{cd}}.$$

If $\{f, g\} \neq 0$, then we can find some a, b, c, d such that $\{x_{ab}, x_{cd}\} \frac{\partial f}{\partial x_{ab}} \frac{\partial g}{\partial x_{cd}} \neq 0$. Notice that

$$\deg(\{x_{ab}, x_{cd}\}) \geq 1, \quad \deg\left(\frac{\partial f}{\partial x_{ab}}\right) \geq \deg(f^{\text{low}}) - 1, \quad \deg\left(\frac{\partial g}{\partial x_{cd}}\right) \geq \deg(g^{\text{low}}) - 1.$$

This verifies (18). We now write g_1 as $g_1^{\text{low}} + H_1$ and g_2 as $g_2^{\text{low}} + H_2$. We rewrite $\{g_1, g_2\}$ as follows:

$$\begin{aligned}\{g_1, g_2\} &= \{g_1^{\text{low}}, g_2^{\text{low}}\} + \{g_1^{\text{low}}, H_2\} + \{H_1, g_2^{\text{low}}\} + \{H_1, H_2\} \\ &= \{g_1^{\text{low}}, g_2^{\text{low}}\} + H \\ &= \{g_1^{\text{low}}, g_2^{\text{low}}\}_0 + h + H.\end{aligned}$$

We see that $h = 0$ or $\deg(h) > \deg(\{g_1^{\text{low}}, g_2^{\text{low}}\}_0)$. By (18), we have either $H = 0$ or $\deg(H) > \deg(\{g_1^{\text{low}}, g_2^{\text{low}}\}_0)$. Thus,

$$\begin{aligned}\{g_1, g_2\}^{\text{low}} &= \{g_1^{\text{low}}, g_2^{\text{low}}\}_0 \\ &= \sum_{a,b,c,d=1}^{n} \{x_{ab}, x_{cd}\}_0 \frac{\partial g_1}{\partial x_{ab}} \frac{\partial g_2}{\partial x_{cd}},\end{aligned}$$

which gives us

$$\deg\left(\{g_1, g_2\}^{\text{low}}\right) = \deg(g_1^{\text{low}}) + \deg(g_2^{\text{low}}) - 1.$$

However, as g_1 and g_2 are log-canonical, $\{g_1, g_2\} = d g_1 g_2$ for some constant d. Thus,

$$\{g_1, g_2\}^{\text{low}} = d g_1^{\text{low}} g_2^{\text{low}},$$

$$\deg\left(\{g_1, g_2\}^{\text{low}}\right) = \deg(g_1^{\text{low}}) + \deg(g_2^{\text{low}}).$$

Hence, we have a contradiction. □

Corollary 3.3. *Given any two of our minors M_i and M_j (where $i, j = 1, \ldots, n^2$) defined in Section 2.2, we have $\{M_i^{low}, M_j^{low}\}_0 = 0$.*

3.2 Poisson commutative functions arising from Casimirs

As will be later shown in Section 4.4.1, the functions $M_1^{\text{low}}, \ldots, M_{n^2}^{\text{low}}$ are not all distinct, and they do not generate a maximal Poisson commutative subalgebra. Thus, we need to construct more Poisson commutative functions in $(\mathcal{R}_n, \{\cdot,\cdot\}_0)$. It is natural for us to consider taking the lowest degree terms of Casimirs in $(\mathcal{R}_n, \{\cdot,\cdot\})$, as they Poisson commute with all functions in $\mathcal{R}_n$.

Definition 3.4. A *Casimir* for a Poisson algebra $(A, \{\cdot,\cdot\})$ is an element $f \in A$ such that $\{f, g\} = 0$ for all $g \in A$. A *complete set* of Casimirs on $(A, \{\cdot,\cdot\})$ is a maximal set of algebraically independent Casimirs.

In [2], Kogan and Zelevinsky implied the following construction for a complete set of Casimirs in $(\mathcal{R}_n, \{\cdot,\cdot\})$:

Theorem 3.5. *The set*

$$\left\{ \frac{\Delta_{[n-i+1,n],[1,i]}(\text{Id}_n + X)}{\Delta_{[1,n-i],[i+1,n]}(\text{Id}_n + X)} : 0 \le i \le n-1 \right\}$$

is a complete set of Casimirs for $(\mathbb{R}(x_{11}, \ldots, x_{nn}), \{\cdot,\cdot\})$, where $\mathbb{R}(x_{11}, \ldots, x_{nn})$ is the fraction field of $\mathcal{R}_n$.

We note that when $i = 0$, we use

$$\Delta_{[n-i+1,n],[1,i]} = 1.$$

As with our minors, we can similarly construct Poisson commutative functions on $(\mathcal{R}_n, \{\cdot,\cdot\}_0)$ from Casimirs on $(\mathcal{R}_n, \{\cdot,\cdot\})$ by taking their lowest degree terms.

Proposition 3.6. *Suppose that c is a Casimir for $(\mathcal{R}_n, \{\cdot,\cdot\})$. Then c^{low} is a Casimir for $(\mathcal{R}_n, \{\cdot,\cdot\}_0)$.*

Proof. It suffices to show that $\{c^{\text{low}}, x_{ij}\}_0 = 0$ for all i and j. As c is a Casimir for $(\mathcal{R}_n, \{\cdot,\cdot\})$, we have $\{c, x_{ij}\} = 0$. Thus, c and x_{ij} are log-canonical in $(\mathcal{R}_n, \{\cdot,\cdot\})$. By Theorem 3.2, we see that $\{c^{\text{low}}, x_{ij}^{\text{low}}\}_0 = 0$. Therefore, $\{c^{\text{low}}, x_{ij}\}_0 = 0$. □

4 Constructing an integrable system for $(\mathcal{R}_n, \{\cdot,\cdot\}_0)$

In the previous section, we showed how to construct commutative functions in $(\mathcal{R}_n, \{\cdot,\cdot\}_0)$ by taking the lowest degree terms of log-canonical systems, which suggests that we may be able to construct an integrable system for $(\mathcal{R}_n, \{\cdot,\cdot\}_0)$. By an integrable system for $(\mathcal{R}_n, \{\cdot,\cdot\}_0)$, we mean

Definition 4.1. Let $\mathbf{F} = (F_1, \dots, F_{\frac{n(n+1)}{2}})$ be a $\frac{n(n+1)}{2}$-tuple of elements in $\mathcal{R}_n$. Then $(\mathcal{R}_n, \{\cdot,\cdot\}_0, \mathbf{F})$ is a *Liouville integrable system* if the following conditions are satisfied:

1. $\mathbf{F}$ is algebraically independent;
2. $\mathbf{F}$ is in involution, *i.e.*, $\{F_i, F_j\}_0 = 0$ for any $1 \le i, j \le \frac{n(n+1)}{2}$.

Remark 4.2. Note that by Theorem 2.9, the rank of $\{\cdot,\cdot\}_0$ is $n^2 - n$. Thus, we need $n^2 - \frac{(n^2-n)}{2} = \frac{n^2+n}{2}$ functions, which is the cardinality of the set $\mathbf{F}$, to generate a maximal Poisson commutative subalgebra for $(\mathcal{R}_n, \{\cdot,\cdot\}_0)$.

In this section, we will give a detailed description of the set $\mathbf{F}$ in Subsection 4.1, and then show that $\mathbf{F}$ do satisfy the conditions in Definition 4.1 in Subsections 4.2 and 4.3. The proofs there are based on calculations shown in Subsection 4.4. These calculations also explain the motivation for our choice of the set $\mathbf{F}$.

4.1 Main Theorem

Recall that our minors $M_1, \dots, M_{n^2}$ are the functions in the following sets:

1. $\{\Delta_{[1,i],[1,i]}(\mathrm{Id}_n + X) : 1 \le i \le n\}$;
2. $\{\Delta_{[k-i+1,k],[1,i]}(\mathrm{Id}_n + X) : 1 \le i < k \le n\}$;
3. $\{\Delta_{[1,i],[k-i+1,k]}(\mathrm{Id}_n + X) : 1 \le i < k \le n\}$.

Observe that if $i \le \lfloor \frac{n}{2} \rfloor$ and $2i \le k \le n$, the minors $\Delta_{[k-i+1,k],[1,i]}(\mathrm{Id}_n + X)$ and $\Delta_{[1,i],[k-i+1,k]}(\mathrm{Id}_n + X)$ do not have any diagonal entries. Thus,

$$(\Delta_{[k-i+1,k],[1,i]}(\mathrm{Id}_n + X))^{\mathrm{low}} = \Delta_{[k-i+1,k],[1,i]}(X), \quad (\Delta_{[1,i],[k-i+1,k]}(\mathrm{Id}_n + X))^{\mathrm{low}} = \Delta_{[1,i],[k-i+1,k]}(X).$$

Let $\mathbf{D}$ be the set of the lowest degree terms of such minors. More explicitly,

$$\mathbf{D} = \left\{\Delta_{[k-i+1,k],[1,i]}(X) : 1 \le i \le \lfloor \frac{n}{2} \rfloor, 2i \le k \le n\right\} \cup \left\{\Delta_{[1,i],[k-i+1,k]}(X) : 1 \le i \le \lfloor \frac{n}{2} \rfloor, 2i \le k \le n\right\}.$$

For conciseness, we will let $\Delta_{I,J}$ denote the minors of matrix X, and $\Delta_{I,J}(\mathrm{Id}_n + X)$ denote the minors of $\mathrm{Id}_n + X$. We will specify the notation again when necessary.

Remark 4.3. We will later show in Subsection 4.3 that $\mathbf{D}$ is, in fact, the set of all nonconstant M_i^{low}'s.

Next, we introduce another set of functions. For each $i = 0, \dots, n-1$, define

$$C_i = \Delta_{[i+1,n],[1,n-i]}(\mathrm{Id}_n + X)\Delta_{[1,n-i],[i+1,n]}(\mathrm{Id}_n + X) - \Delta_{[n-i+1,n],[1,i]}(\mathrm{Id}_n + X)\Delta_{[1,i],[n-i+1,n]}(\mathrm{Id}_n + X),$$

and denote the set

$$\mathbf{C} = \left\{C_i^{\mathrm{low}} : 0 \le i \le \lfloor \frac{n-1}{2} \rfloor\right\}.$$

We will explain the motivation for this construction in Subsection 4.3 and show the explicit expression for C_i^{low} in Proposition 4.10.

We are now ready to propose our main theorem:

Theorem 4.4. *Let $\mathbf{F} = \mathbf{C} \cup \mathbf{D}$ as above. Then $(\mathcal{R}_n, \{\cdot,\cdot\}_0, \mathbf{F})$ is a Liouville integrable system.*

We can easily verify that the number of functions in $\mathbf{F}$ is indeed $\frac{n^2+n}{2}$. We will split the proof of Theorem 4.4 into two steps: first, we will verify that $\mathbf{F}$ is algebraically independent; then, we will show that all functions in $\mathbf{F}$ Poisson commute with each other.

4.2 F is algebraically independent

In this part, we show that **F** is algebraically independent by examining its Jacobian matrix.

Definition 4.5. An s-tuple of functions $(f_1, \dots, f_s)$ in $\mathbb{R}[x_1, \cdots, x_d]$ is algebraically independent if its Jacobian matrix J, given by

$$J_{ij} = \left(\frac{\partial f_i}{\partial x_j}\right),$$

has full rank.

We see that the Jacobian matrix J for **F** is a $(\frac{n^2+n}{2}) \times n^2$ matrix. We will therefore show that

Proposition 4.6. $\operatorname{rank}(J) = \frac{n^2+n}{2}$.

It suffices for us to find a $(\frac{n^2+n}{2}) \times (\frac{n^2+n}{2})$ submatrix of J whose diagonal entries are all nonzero and is lower-triangular. We will fix a specific order for $\{x_{i,j}\}$ and the functions in **F** such that S, which denotes the submatrix $J_{[1,\frac{n^2+n}{2}],[1,\frac{n^2+n}{2}]}$, satisfies the following:

(I) $S_{ii} \neq 0$;

(II) $S_{ij} = 0$ for all $j > i$.

Proof of Proposition 4.6. When n is even, we let the first $\frac{n^2+n}{2}$ terms of $\{x_{i,j}\}$ be the sequence

$$x = (x_{1,n}, \dots, x_{1,2}, x_{n,1}, \dots, x_{2,1}, x_{2,n-1}, \dots, x_{2,3}, x_{n-1,2}, \dots, x_{3,2}, \dots, x_{\frac{n}{2},\frac{n}{2}+1}, x_{\frac{n}{2}+1,\frac{n}{2}}, x_{\frac{n}{2},\frac{n}{2}}, \dots, x_{2,2}, x_{1,1}).$$

We let the (first) $\frac{n^2+n}{2}$ terms of **F** be the sequence

$$F = (\Delta_{[1],[n]}, \dots, \Delta_{[1],[2]}, \Delta_{[n],[1]}, \dots, \Delta_{[2],[1]}, \Delta_{[1,2],[n-1,n]}, \dots, \Delta_{[1,2],[3,4]}, \Delta_{[n-1,n],[1,2]}, \dots, \Delta_{[3,4],[1,2]}, \dots,$$
$$\Delta_{[1,\frac{n}{2}],[\frac{n}{2}+1,n]}, \Delta_{[\frac{n}{2}+1,n],[1,\frac{n}{2}]}, C^{\text{low}}_{\frac{n}{2}-1}, \dots, C^{\text{low}}_1, C^{\text{low}}_0).$$

When n is odd, we let the first $\frac{n^2+n}{2}$ terms of $\{x_{i,j}\}$ be

$$x = (x_{1,n}, \dots, x_{1,2}, x_{n,1}, \dots, x_{2,1}, x_{2,n-1}, \dots, x_{2,3}, x_{n-1,2}, \dots, x_{3,2}, \dots,$$
$$x_{\lfloor\frac{n}{2}\rfloor,\lfloor\frac{n}{2}\rfloor+2}, x_{\lfloor\frac{n}{2}\rfloor,\lfloor\frac{n}{2}\rfloor+1}, x_{\lfloor\frac{n}{2}\rfloor+2,\lfloor\frac{n}{2}\rfloor}, x_{\lfloor\frac{n}{2}\rfloor+1,\lfloor\frac{n}{2}\rfloor}, x_{\lfloor\frac{n}{2}\rfloor+1,\lfloor\frac{n}{2}\rfloor+2}, x_{\lfloor\frac{n}{2}\rfloor,\lfloor\frac{n}{2}\rfloor}, \dots, x_{2,2}, x_{1,1})$$

We let the first $\frac{n^2+n}{2}$ terms of **F** be

$$F = (\Delta_{[1],[n]}, \dots, \Delta_{[1],[2]}, \Delta_{[n],[1]}, \dots, \Delta_{[2],[1]}, \Delta_{[1,2],[n-1,n]}, \dots, \Delta_{[1,2],[3,4]}, \Delta_{[n-1,n],[1,2]}, \dots, \Delta_{[3,4],[1,2]}, \dots,$$
$$\Delta_{[1,\lfloor\frac{n}{2}\rfloor],[\lfloor\frac{n}{2}\rfloor+2,n]}, \Delta_{[1,\lfloor\frac{n}{2}\rfloor],[\lfloor\frac{n}{2}\rfloor+1,n-1]}, \Delta_{[\lfloor\frac{n}{2}\rfloor+2,n],[1,\lfloor\frac{n}{2}\rfloor]}, \Delta_{[\lfloor\frac{n}{2}\rfloor+1,n-1],[1,\lfloor\frac{n}{2}\rfloor]}, C^{\text{low}}_{\lfloor\frac{n}{2}\rfloor}, \dots, C^{\text{low}}_1, C^{\text{low}}_0).$$

We will use the expression for C^{low}_i given in Proposition 4.10: for all $1 \leq i \leq \lfloor\frac{n-1}{2}\rfloor$,

$$C^{\text{low}}_i = \sum_{k\in[i+1,n-i]} \Delta_{[n-i+1,n],[1,i]}(X)\Delta_{k\cup[1,i],k\cup[n-i+1,n]}(X) + \Delta_{[1,i],[n-i+1,n]}(X)\Delta_{k\cup[n-i+1,n],k\cup[1,i]}(X), \tag{19}$$

and for $i = 0$,

$$C^{\text{low}}_i = 2\operatorname{tr}(X). \tag{20}$$

We will show that for a fixed i, F_i has x_i in its expansion, and for every $k > i$, F_i does not have x_j in its expansion. We will do this in two steps.

(1) $1 \leq i \leq \frac{n^2}{2}$ (or $1 \leq i \leq \frac{n^2-1}{2}$ for odd n).
We have either

$$x_i = x_{a,n-a+1}, \quad F_i = \Delta_{[1,a],[n-a+1,n]},$$

or

$$x_i = x_{n-a+1,a}, \quad F_i = \Delta_{[n-a+1,n],[1,a]}.$$

Without loss of generality, we will only consider the first case.

We see that x_i is the bottom left entry of the submatrix $A'_{[1,a],[n-a+1,n]}$. Thus, x_i is in the expansion of F_i.

For every $j > i$, we have three possible cases:

(i) $x_j = x_{b,n-b+1}$, where $a < b \leq \frac{n}{2} + 1$ (or $a < b \leq \lfloor \frac{n}{2} \rfloor + 2$ for odd n).
(ii) $x_j = x_{n-b+1,b}$, where $a \leq b \leq \frac{n}{2} + 1$ (or $a \leq b \leq \lfloor \frac{n}{2} \rfloor + 2$ for odd n).
(iii) $x_j = x_{m,m}$, where $1 \leq m \leq \lfloor \frac{n}{2} \rfloor$.

We can easily observe that F_i does not have x_j in its expansion in each case.

(2) $\frac{n^2}{2} + 1 \leq i \leq \frac{n^2+n}{2}$ (or $\frac{n^2+1}{2} \leq i \leq \frac{n^2+n}{2}$ for odd n).
In this case, we have $x_i = x_{m,m}$, $F_i = C^{\text{low}}_{m-1}$, where $m = \frac{n^2+n}{2} - i + 1$.

When $i = \frac{n^2+n}{2}$, we have $F_i = C^{\text{low}}_0 = 2\,\text{tr}(X)$, so x_i appears in F_i.

For all other i, recall that

$$C^{\text{low}}_{m-1} = \sum_{k \in [m,n-m+1]} (\Delta_{[n-m,n],[1,m-1]} \Delta_{k \cup [1,m-1],k \cup [n-m,n]} + \Delta_{[1,m-1],[n-m,n]} \Delta_{k \cup [n-m,n],k \cup [1,m-1]}).$$

For a fixed i, x_i appears in the sum where $k = m$, so x_i is in the expansion of F_i.

For every $j > i$, we have $\frac{n^2+n}{2} - j + 1 < m$. Notice also that $1 \leq m \leq \frac{n}{2}$, so $n - m \geq m > \frac{n^2+n}{2} - j + 1$. Thus, x_j is not in $\Delta_{k \cup [1,m-1],k \cup [n-m,n]}$ or $\Delta_{k \cup [n-m,n],k \cup [1,m-1]}$ for all k. Moreover, x_j is also not in $\Delta_{[n-m,n],[1,m-1]}$ and $\Delta_{[1,m-1],[n-m,n]}$. Hence, F_i does not have x_j in its expansion.

When n is odd, we consider the special case when $i = \frac{n^2+1}{2}$, that is,

$$\begin{aligned} x_i &= x_{\lfloor \frac{n}{2} \rfloor + 1, \lfloor \frac{n}{2} \rfloor + 2} \\ F_i &= C^{\text{low}}_{\lfloor \frac{n}{2} \rfloor} \\ &= \sum_{k \in [l+1,n-l]} (\Delta_{[n-l+1,n],[1,l]} \Delta_{k \cup [1,l],k \cup [n-l+1,n]} + \Delta_{[1,l],[n-l+1,n]} \Delta_{k \cup [n-l+1,n],k \cup [1,l]}), \end{aligned}$$

where $l = \lfloor \frac{n}{2} \rfloor$. We see that $l + 1 = n - l = \frac{n+1}{2}$, so

$$F_i = \Delta_{[n-l+1,n],[1,l]} \Delta_{[1,l+1],[n-l,n]} + \Delta_{[1,l],[n-l+1,n]} \Delta_{[n-l,n],[1,l+1]}$$

We see that x_i appears in $\Delta_{[1,l+1],[n-l,n]}$. For every $j > i$, $x_j = x_{a,a}$, where $a \leq \lfloor \frac{n}{2} \rfloor$. Thus, x_j does not appear in the expansion of F_i. □

Example 4.7. In $(\mathcal{R}_5, \{\cdot,\cdot\}_0)$, we fix the first 15 terms of $\{x_{ij}\}$ and $\mathbf{F}$ to be the sequences

$$\begin{aligned} x &= (x_{15}, x_{14}, x_{13}, x_{12}, x_{51}, x_{41}, x_{31}, x_{21}, x_{24}, x_{23}, x_{42}, x_{32}, x_{34}, x_{22}, x_{11}) \\ F &= (\Delta_{[1],[5]}, \Delta_{[1],[4]}, \Delta_{[1],[3]}, \Delta_{[1],[2]}, \Delta_{[5],[1]}, \Delta_{[4],[1]}, \Delta_{[3],[1]}, \Delta_{[2],[1]}, \Delta_{[1,2],[4,5]}, \Delta_{[1,2],[3,4]}, \Delta_{[4,5],[1,2]}, \Delta_{[3,4],[1,2]}, \\ &\quad C^{\text{low}}_2, C^{\text{low}}_1, C^{\text{low}}_0). \end{aligned}$$

The 15×15 submatrix S that we obtained from calculation is

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & 0 & 0 & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & * & * & 0 & 0 & 0 & 0 & 0 & * & * & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & * & * & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & * & * & 0 & 0 & 0 & * & * & 0 & 0 & 0 \\
* & * & * & 0 & * & * & * & 0 & * & * & * & * & * & 0 & 0 \\
* & * & * & * & * & * & * & * & 0 & 0 & 0 & 0 & 0 & * & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2
\end{pmatrix},$$

where $*$ denotes a nonzero entry.

4.3 F is in involution

Proof of Theorem 4.4. All we need to show now is that $\mathbf{F}$ is in involution.

Recall that

$$\left\{ \frac{\Delta_{[n-i+1,n],[1,i]}(\mathrm{Id}_n + X)}{\Delta_{[1,n-i],[i+1,n]}(\mathrm{Id}_n + X)} : 0 \le i \le n-1 \right\}$$

is a complete set of Casimirs for $\{\cdot,\cdot\}$. However, by Proposition 4.8, we have

$$\left(\frac{\Delta_{[n-i+1,n],[1,i]}(\mathrm{Id}_n + X)}{\Delta_{[1,n-i],[i+1,n]}(\mathrm{Id}_n + X)} \right)^{\mathrm{low}} = \left(\frac{\Delta_{[i+1,n],[1,n-i]}(\mathrm{Id}_n + X)}{\Delta_{[1,i],[n-i+1,n]}(\mathrm{Id}_n + X)} \right)^{\mathrm{low}} = \frac{M_j^{\mathrm{low}}}{M_l^{\mathrm{low}}}$$

for some j and l.

As Casimirs under linear combinations are still Casimirs, it is natural for us to consider the following set of Casimirs:

$$\widehat{\mathbf{C}} := \left\{ \frac{\Delta_{[n-i+1,n],[1,i]}(\mathrm{Id}_n + X)}{\Delta_{[1,n-i],[i+1,n]}(\mathrm{Id}_n + X)} - \frac{\Delta_{[i+1,n],[1,n-i]}(\mathrm{Id}_n + X)}{\Delta_{[1,i],[n-i+1,n]}(\mathrm{Id}_n + X)} : 0 \le i \le \lfloor \frac{n-1}{2} \rfloor \right\}. \tag{21}$$

Now let us consider the set

$$\{M_1, \ldots, M_{n^2}\} \cup \widehat{\mathbf{C}}.$$

By Corollary 3.3 and Proposition 3.6, we know that the functions in

$$\{M_1, \ldots, M_{n^2}\}^{\mathrm{low}} \cup \widehat{\mathbf{C}}^{\mathrm{low}}$$

Poisson commute with each other under the linear bracket$\{\cdot,\cdot\}_0$.

By Proposition 4.8, we know that $\mathbf{D}$ is the set of all nonconstant distinct functions in $\{M_i^{\mathrm{low}}\}$. It is easy to see that our set $\mathbf{C}$ comes from taking the lowest degree terms of the numerators of functions in $\widehat{\mathbf{C}}$. As the lowest degree terms of the denominators of functions in $\widehat{\mathbf{C}}$ appear in the set $\mathbf{D}$, all functions in $\mathbf{F}$ do indeed Poisson commute with each other. □

4.4 Motivation for choice of functions

In this part, we will explain our choice of functions in $\mathbf{C}$ and $\mathbf{D}$. We first compute the explicit expressions for $M_1^{\mathrm{low}}, \ldots, M_{n^2}^{\mathrm{low}}$ to show that $\mathbf{D}$ is in fact the set of all nonconstant distinct functions in $\{M_i^{\mathrm{low}}\}$ and explain why we have to construct an additional set $\mathbf{C}$ for our integrable system. We will then use observations from our computation of $M_1^{\mathrm{low}}, \ldots, M_{n^2}^{\mathrm{low}}$ to give the explicit expression for C_i^{low}.

4.4.1 The explicit expressions for the lowest degree terms of minors

Recall again that the minor M_s, where $s = 1, \dots, n^2$, must be one of the following forms:

1. $\Delta_{[1,i],[1,i]}(\mathrm{Id}_n + X)$, where $1 \le i \le n$;
2. $\Delta_{[k-i+1,k],[1,i]}(\mathrm{Id}_n + X)$, where $1 \le i < k \le n$;
3. $\Delta_{[1,i],[k-i+1,k]}(\mathrm{Id}_n + X)$, where $1 \le i < k \le n$.

Clearly, $(\Delta_{[1,i],[1,i]}(\mathrm{Id}_n + X))^{\mathrm{low}} = 1$. We will now compute the explicit expression for $(\Delta_{[k-i+1,k],[1,i]}(\mathrm{Id}_n + X))^{\mathrm{low}}$.

Proposition 4.8. *Suppose that $M_s = \Delta_{[k-i+1,k],[1,i]}(\mathrm{Id}_n + X)$. Then*

$$M_s^{low} = \begin{cases} \Delta_{[k-i+1,k],[1,i]}(X), & \text{if } i \le \lfloor \frac{n}{2} \rfloor \text{ and } 2i \le k \le n; \\ \Delta_{[i+1,k],[1,k-i]}(X), & \text{if } i \ge \lfloor \frac{n}{2} \rfloor \text{ and } k > i, \text{ or } i < k < 2i. \end{cases}$$

Proof. We will show this in two steps.

(i) Suppose that $i \le \lfloor \frac{n}{2} \rfloor$ and $2i \le k \le n$.
Then $[k-i+1, k] \cap [1, i] = \emptyset$. Thus, there are no diagonal entries of the matrix $\mathrm{Id}_n + X$ containing in $\Delta_{[k-i+1,k],[1,i]}(\mathrm{Id}_n + X)$, so it is still a homogeneous polynomial when written in the coordinates $\{x_{ij}\}$. Hence,

$$(\Delta_{[k-i+1,k],[1,i]}(\mathrm{Id}_n + X))^{\mathrm{low}} = \Delta_{[k-i+1,k],[1,i]}(X).$$

(ii) Suppose that $i \ge \lfloor \frac{n}{2} \rfloor$ (and $k > i$), or $i < k < 2i$.
Then $\Delta_{[k-i+1,k],[1,i]}(\mathrm{Id}_n + X)$ has diagonal entries. To visualise, it is the determinant of the following matrix:

$$\left(\begin{array}{c|ccc} & x_{k-i+1,k-i+1}+1 & & \\ R & & \ddots & \\ & & & x_{i,i}+1 \\ \hline T & & B & \end{array}\right),$$

where R, B, T are the corresponding block submatrices of X.

Because of the elements $x_{r,r} + 1$, where $r = k-i+1, \dots, i$, the polynomial $\Delta_{[k-i+1,k],[1,i]}(\mathrm{Id}_n + X)$ is not homogeneous. All monomials in $\Delta_{[k-i+1,k],[1,i]}(\mathrm{Id}_n + X)$ have degree i, except the term

$$\prod_{m \in [k-i+1,i]} (x_{m,m} + 1) \cdot \Delta_{[i+1,k],[1,k-i]}(\mathrm{Id}_n + X)$$

which contributes the lowest degree terms of $\Delta_{[i+1,k],[1,k-i]}(\mathrm{Id}_n + X)$. That is,

$$(\Delta_{[k-i+1,k],[1,i]}(\mathrm{Id}_n + X))^{\mathrm{low}} = \left(\prod_{m \in [k-i+1,i]} (x_{m,m} + 1) \cdot \Delta_{[i+1,k],[1,k-i]}(\mathrm{Id}_n + X) \right)^{\mathrm{low}}. \tag{22}$$

Simplifying, we have that

$$(\Delta_{[k-i+1,k],[1,i]}(\mathrm{Id}_n + X))^{\mathrm{low}} = (\Delta_{[i+1,k],[1,k-i]}(\mathrm{Id}_n + X))^{\mathrm{low}}.$$

We see that $1 \le k - i \le \lfloor \frac{n}{2} \rfloor$ and $2(k-i) \le k \le n$. Thus, by our result in part (i), we have that

$$(\Delta_{[i+1,k],[1,k-i]}(\mathrm{Id}_n + X))^{\mathrm{low}} = \Delta_{[i+1,k],[1,k-i]}(X).$$

Hence,

$$(\Delta_{[k-i+1,k],[1,i]}(\mathrm{Id}_n + X)^{\mathrm{low}} = \Delta_{[i+1,k],[1,k-i]}(X).$$ □

We stress again that we write $\Delta_{I,J}$ for the minor of the matrix X (not $\mathrm{Id}_n + X$) for simplicity.

Example 4.9. The nonconstant minor lows in $(\mathcal{R}_3, \{\cdot,\cdot\}_0)$ are

$$\Delta_{[2],[1]}, \Delta_{[3],[1]}, \Delta_{[1],[2]}, \Delta_{[1],[3]}.$$

The nonconstant minor lows in $(\mathcal{R}_5, \{\cdot,\cdot\}_0)$ are

$$\Delta_{[2],[1]}, \Delta_{[3],[1]}, \Delta_{[4],[1]}, \Delta_{[5],[1]}, \Delta_{[4,5],[1,2]}, \Delta_{[3,4],[1,2]}, \Delta_{[1],[2]}, \Delta_{[1],[3]}, \Delta_{[1],[4]}, \Delta_{[1],[5]}, \Delta_{[1,2],[4,5]}, \Delta_{[1,2],[3,4]}.$$

It follows from Proposition 4.8 that $\mathbf{D}$ is the set of distinct, nonconstant M_s^{low}s. As was noted earlier in Section 3.2, there are only $\lfloor \frac{n^2}{2} \rfloor$ functions in $\mathbf{D}$, which is not enough to construct an integrable system of $(\mathcal{R}_n, \{\cdot,\cdot\}_0)$. Thus, we had to construct the remaining functions using Casimirs.

4.4.2 The explicit expression for C_i^{low}

In this section, we complete the proof of Theorem 4.4 by proving the expressions for C_i^{low} given in (19) and (20).

Proposition 4.10. *For $i = 0$, we have that*

$$C_i^{low} = 2\,\mathrm{tr}(X).$$

For every $1 \le i \le \lfloor \frac{n-1}{2} \rfloor$, we have that

$$C_i^{low} = \sum_{k \in [i+1, n-i]} (\Delta_{[n-i+1,n],[1,i]} \Delta_{k \cup [1,i], k \cup [n-i+1,n]} + \Delta_{[1,i],[n-i+1,n]} \Delta_{k \cup [n-i+1,n], k \cup [1,i]}).$$

Again, we use $\Delta_{I,J}$ to denote the minor of matrix X, and write $\Delta_{I,J}(\mathrm{Id}_n + X)$ for the minor of matrix $\mathrm{Id}_n + X$.

Proof of Proposition 4.10. We prove each expression separately.

1. Suppose that $i = 0$. We see that

$$\begin{aligned} C_0 &= \Delta_{[1,n],[1,n]}(\mathrm{Id}_n + X) \cdot \Delta_{[1,n],[1,n]}(\mathrm{Id}_n + X) - 1 \\ &= (\det(\mathrm{Id}_n + X) + 1)(\det(\mathrm{Id}_n + X) - 1). \end{aligned}$$

   It is easy to observe that

$$\det(\mathrm{Id}_n + X) = H + \prod_{j=1}^{n} (x_{jj} + 1),$$

   where $\deg(H^{\text{low}}) \ge 2$. Thus, $(\det(\mathrm{Id}_n + X))^{\text{low}} = 1$, so $(\det(\mathrm{Id}_n + X) + 1)^{\text{low}} = 2$. We also see that

$$\begin{aligned} (\det(\mathrm{Id}_n + X) - 1)^{\text{low}} &= \sum_{j=1}^{n} x_{jj} \\ &= \mathrm{tr}(X). \end{aligned}$$

   For any two functions f and g, we have $(fg)^{\text{low}} = f^{\text{low}} g^{\text{low}}$, so it follows that $C_0 = 2\,\mathrm{tr}(X)$.

2. For conciseness, we will let

$$D_1 = \Delta_{[1,n-i],[i+1,n]}, \quad D_2 = \Delta_{[1,i],[n-i+1,n]}.$$

   Notice that

$$D_1^{\intercal} = \Delta_{[i+1,n],[1,n-i]}, \quad D_2^{\intercal} = \Delta_{[n-i+1,n],[1,i]},$$

where $D_1^\intercal$ denotes $\Delta_{[1,n-i],[i+1,n]}(X^\intercal)$ and $D_2^\intercal$ denotes $\Delta_{[1,i],[n-i+1,n]}(X^\intercal)$. Thus,

$$C_i^{\text{low}} = (D_1^\intercal D_1 - D_2^\intercal D_2)^{\text{low}}.$$

It follows from our proof of Proposition 4.8 that $D_1^{\text{low}} = D_2, D_1^{\intercal\text{low}} = D_2^\intercal, D_2^{\text{low}} = D_2$, and $D_2^{\intercal\text{low}} = D_2^\intercal$. Writing D_1 as $D_2 + P$, and $D_1^\intercal$ as $D_2^\intercal + Q$, we have

$$\begin{aligned} C_i^{\text{low}} &= ((D_2 + P)(D_2^\intercal + Q) - D_2^\intercal D_2)^{\text{low}} \\ &= (D_2 Q + P D_2^\intercal + PQ)^{\text{low}}. \end{aligned}$$

Observe that $\deg((PQ)^{\text{low}}) > \deg(D_2 Q), \deg(PD_2^\intercal)$. Thus,

$$C_i^{\text{low}} = (D_2 Q + P D_2^\intercal)^{\text{low}}.$$

We will now show that $(D_2 Q + P D_2^\intercal)^{\text{low}} = (D_2 Q)^{\text{low}} + (P D_2^\intercal)^{\text{low}}$.

Notice that $(D_2 Q)^{\text{low}} = D_2 Q^{\text{low}}$ and $(PD_2^\intercal)^{\text{low}} = P^{\text{low}} D_2^\intercal$ have the same degree. It is easy to see that, for any two polynomials f and g such that $\deg(f^{\text{low}}) = \deg(g^{\text{low}})$, we have $(f+g)^{\text{low}} = f^{\text{low}} + g^{\text{low}}$ if and only if $f^{\text{low}} + g^{\text{low}} \neq 0$.

Recall again from the proof of Proposition 4.8 that

$$D_1^{\text{low}} = \left(\prod_{m \in [i+1, n-i]} (x_{m,m} + 1) \cdot \Delta_{[1,i],[n-i+1,n]} \right)^{\text{low}}.$$

As the terms in P^{low} have the second lowest order in D_1, they are of the form

$$\left(\prod_{m \in [i+1, n-i] \setminus k} (x_{m,m} + 1) \cdot \Delta_{k \cup [1,i], k \cup [n-i+1,n]} \right)^{\text{low}}$$

where $k \in [i+1, n-i]$. Thus,

$$P^{\text{low}} = \sum_{k \in [i+1, n-i]} \Delta_{k \cup [1,i], k \cup [n-i+1,n]}.$$

By symmetry, we also have that

$$Q^{\text{low}} = \sum_{k \in [i+1, n-i]} \Delta_{k \cup [n-i+1,n], k \cup [1,i]}.$$

Therefore,

$$(D_2 Q)^{\text{low}} + (D_2^\intercal P)^{\text{low}} = \sum_{k \in [i+1, n-i]} \Delta_{[n-i+1,n],[1,i]} \Delta_{k \cup [1,i], k \cup [n-i+1,n]} + \Delta_{[1,i],[n-i+1,n]} \Delta_{k \cup [n-i+1,n], k \cup [1,i]},$$

which is clearly nonzero. Thus, $(D_2 Q + P D_2^\intercal)^{\text{low}} = (D_2 Q)^{\text{low}} + (P D_2^\intercal)^{\text{low}}$. Therefore,

$$\begin{aligned} C_i^{\text{low}} &= (D_2 Q + P D_2^\intercal)^{\text{low}} \\ &= (D_2 Q)^{\text{low}} + (P D_2^\intercal)^{\text{low}} \\ &= \sum_{k \in [i+1, n-i]} (\Delta_{[n-i+1,n],[1,i]} \Delta_{k \cup [1,i], k \cup [n-i+1,n]} + \Delta_{[1,i],[n-i+1,n]} \Delta_{k \cup [n-i+1,n], k \cup [1,i]}). \end{aligned}$$

□

Example 4.11. For $(\mathcal{R}_3, \{\cdot,\cdot\}_0)$, the functions in $\mathbf{C}$ are

$$C_0^{\text{low}} = 2x_{11} + 2x_{22} + 2x_{33},$$

$$C_1^{\text{low}} = \Delta_{[3],[1]}\Delta_{[1,2],[2,3]} + \Delta_{[1],[3]}\Delta_{[2,3],[1,2]}.$$

For $(\mathcal{R}_5, \{\cdot,\cdot\}_0)$, the functions in $\mathbf{C}$ are

$$C_0^{\text{low}} = 2x_{11} + 2x_{22} + 2x_{33} + 2x_{44} + 2x_{55},$$

$$C_1^{\text{low}} = \sum_{k\in[2,4]} (\Delta_{[5],[1]}\Delta_{k\cup[1],k\cup[5]} + \Delta_{[1],[5]}\Delta_{k\cup[5],k\cup[1]}),$$

$$C_2^{\text{low}} = \Delta_{[4,5],[1,2]}\Delta_{[1,3],[3,5]} + \Delta_{[1,2],[4,5]}\Delta_{[3,5],[1,3]}.$$

5 Quantizing the Poisson commutative functions

In this section, we study the quantum integrable system, *i.e.*, a maximally commutative subalgebra of the universal enveloping algebra $U(\mathfrak{p})$, corresponding to the classical Liouville integrable system that we constructed in Theorem 4.4. In particular, we solve the quantization problem for the minors in $\mathbf{D}$ by constructing the set $q\mathbf{D}$ of quantum minors and showing that they commute with each other.

As mentioned in Section 1.2, to complete the construction of our quantum integrable system, we must also find a set $q\mathbf{C}$ of $\lfloor \frac{n+1}{2} \rfloor$ elements in $U(\mathfrak{p})$ satisfying the following: all elements from $q\mathbf{D} \cup q\mathbf{C}$ commute with each other, and the image of $q\mathbf{C}$ under the isomorphism $\text{gr}(U(\mathfrak{p})) \cong \text{Sym}(\mathfrak{p})$ coincide with $\mathbf{C}$. We have a candidate for the set $q\mathbf{C}$ but did not have enough time to fully verify it, so we will leave it for our future study.

5.1 Commutative subalgebras via the quantum minors

Recall from Section 1.2 that the universal enveloping algebra $U(\mathfrak{p})$ is generated by the generators $\{e_{ij}\}_{i,j=1,\ldots,n}$ with relations (6)-(8). Moreover, the algebra $U(\mathfrak{p})$ is equipped with a canonical filtration and the associated graded algebra $\text{gr}(U(\mathfrak{p}))$ is isomorphic to the symmetric algebra $\text{Sym}(\mathfrak{p})$.

Recall also that we let E be an $n \times n$ matrix with entries valued in $U(\mathfrak{p})$

$$(E)_{ij} = e_{ij}, \qquad \text{for } 1 \le i,j \le n, \quad 1 \le m \le k.$$

Definition 5.1. For any $1 \le i \le \lfloor \frac{n}{2} \rfloor$, $2i \le k \le n$, we define the quantum minors as follows:

$$q\Delta_{[1,i],[k-i+1,k]}(E) := \sum_{\sigma} (-1)^{|\sigma|} e_{1\sigma(1)} e_{2\sigma(2)} \cdots e_{i\sigma(i)} \in U(\mathfrak{p}), \tag{23}$$

$$q\Delta_{[k-i+1,k],[1,i]}(E) := \sum_{\sigma} (-1)^{|\sigma|} e_{\sigma(1)1} e_{\sigma(2)2} \cdots e_{\sigma(i)i} \in U(\mathfrak{p}). \tag{24}$$

Here, the summation is over all the permutations σ of $\{k-i+1, k-i, \ldots, k\}$ and $|\sigma|$ denotes the signature of the permutation σ. We define $q\mathbf{D}$ to be the set of these quantum minors.

We remark that all elements $\{e_{ab}\}_{a=1,\ldots,i,\, b=k-i+1,\ldots,k}$ appearing in (23), commute with each other. Thus, $q\Delta_{[1,i],[k-i+1,k]}(E)$ equals the corresponding column expansion minor, *i.e.*,

$$q\Delta_{[1,i],[k-i+1,k]}(E) = \sum_{\sigma} (-1)^{|\sigma|} e_{\sigma(k-i+1)k-i+1} e_{\sigma(k-i)k-i} \cdots e_{\sigma(i)k}.$$

Furthermore, the Laplace expansion applies to our case. That is,

Lemma 5.1.1. *We can expand* $q\Delta_{[1,i],[k-i+1,k]}(E)$ *using the Laplace column and row expansion, i.e.,*

$$\text{for any integer } b,\ \ q\Delta_{[1,i],[k-i+1,k]}(E) = \sum_{c\in[1,i]} (-1)^{b+c} e_{c,b} \cdot q\Delta_{[1,i]\setminus c,[k-i+1,k]\setminus b}, \tag{25}$$

$$\text{for any integer } c,\ \ q\Delta_{[1,i],[k-i+1,k]}(E) = \sum_{b\in[k-i+1,k]} (-1)^{c+b} e_{c,b} \cdot q\Delta_{[1,i]\setminus c,[k-i+1,k]\setminus b}. \tag{26}$$

Note that for conciseness, we sometimes denote the minors of E by $q\Delta_{[1,i],[k-i+1,k]}$ and $q\Delta_{[1,i],[k-i+1,k]}$.

Notice that $q\Delta_{[1,i],[k-i+1,k]}$ is an element of $U_i(\mathfrak{p}) \subset U(\mathfrak{p})$ with filtered degree i, and its associated graded element coincides with $\Delta_{[1,i],[k-i+1,k]}$. That is,

$$\operatorname{gr}\left(q\Delta_{[1,i],[k-i+1,k]}(E)\right) = \Delta_{[1,i],[k-i+1,k]}(X) \in \operatorname{Sym}(\mathfrak{p}).$$

We will now prove our second main theorem:

Theorem 5.2. *All quantum minors* $q\Delta_{[1,i],[k-i+1,k]}, q\Delta_{[1,i],[k-i+1,k]} \in U(\mathfrak{p})$ *of* E*, where* $1 \le i \le \lfloor \frac{n}{2} \rfloor, 2i \le k \le n$ *commute with each other.*

Proof. Recall that the quantum minors are of the form $q\Delta_{[1,i],[k-i+1,k]}$ or $q\Delta_{[k-i+1,k],[1,i]}$, where $1 \le i \le \lfloor \frac{n}{2} \rfloor$ and $2i \le k \le n$. Thus, every quantum minor is either in the upper triangle or lower triangle of E.

It is trivially true that a quantum minor in the upper triangle of E commutes with a quantum minor in the lower triangle.

We now consider the commutator between two quantum minors in the upper triangle of E. We shall denote them as $q\Delta_{[1,i],[k-i+1,k]}$ and $q\Delta_{[1,j],[m-j+1,m]}$. Assume without loss of generality that $i \le j$. This implies that $i < m-j+1$, so $[1,i] \cap [m-j+1,m] = \emptyset$. If we also have $[k-i+1,k] \cap [1,j] = \emptyset$, then the two quantum minors clearly commute. Thus, we suppose that $[k-i+1,k] \cap [1,j] \ne \emptyset$.

As $j \le \lfloor \frac{m}{2} \rfloor$, we must have $[1,j] \cap [m-j+1,m] = \emptyset$. Thus, all entries e_{ab}, where $a \in [1,j]$ and $b \in [m-j+1,m]$, commute with each other. This means that we can expand $q\Delta_{[1,j],[m-j+1,m]}$ along any row or column (using the Laplace expansion in Lemma 5.1.1). Similarly, we can expand $q\Delta_{[1,i],[k-i+1,k]}$ along any row or column.

To help with intuition, we will first prove the case when $|[k-i+1,k] \cap [1,j]| = 1$. This means that $j = k-i+1$, so

$$q\Delta_{[1,j],[m-j+1,m]} = q\Delta_{[1,k-i+1],[m-k+i,m]}.$$

To manipulate the expansion of $[q\Delta_{[1,i],[k-i+1,k]}, q\Delta_{[1,k-i+1],[m-k+i,m]}]$, we shall use the property that

$$\begin{aligned} [x, yz] &= xyz - yzx \\ &= xyz - yxz + yxz - yzx \\ &= [x,y]z + y[x,z]. \end{aligned} \tag{27}$$

Using the Laplace expansion of $q\Delta_{[1,j],[m-j+1,m]}$ along the $k-i+1^{\text{th}}$ row, we have that

$$\begin{aligned} &[q\Delta_{[1,i],[k-i+1,k]}, q\Delta_{[1,k-i+1],[m-k+i,m]}] \\ &= \left[q\Delta_{[1,i],[k-i+1,k]}, \sum_{b\in[m-k+i,m]} (-1)^{k-i+1+b} e_{k-i+1,b} \cdot q\Delta_{[1,k-i],[m-k+i,m]\setminus b} \right] \\ &= \sum_{b\in[m-k+i,m]} [q\Delta_{[1,i],[k-i+1,k]}, (-1)^{k-i+1+b} e_{k-i+1,b} \cdot q\Delta_{[1,k-i],[m-k+i,m]\setminus b}] \\ &= \sum_{b\in[m-k+i,m]} [q\Delta_{[1,i],[k-i+1,k]}, (-1)^{k-i+1+b} e_{k-i+1,b}] \cdot q\Delta_{[1,k-i],[m-k+i,m]\setminus b} \\ &\quad + \sum_{b\in[m-k+i,m]} (-1)^{k-i+1+b} e_{k-i+1,b} \cdot [q\Delta_{[1,i],[k-i+1,k]}, q\Delta_{[1,k-i],[m-k+i,m]\setminus b}] \end{aligned}$$

Notice that $[k-i+1,k] \cap [1,k-i] = \emptyset$. For every value of b, we also have $[1,i] \cap [m-k+i,m] \backslash b = \emptyset$, so

$$[q\Delta_{[1,i],[k-i+1,k]}, q\Delta_{[1,k-i],[m-k+i,m]\backslash b}] = 0.$$

Now, it suffices for us to show that

Lemma 5.2.1.

$$\sum_{b\in[m-k+i,m]} [q\Delta_{[1,i],[k-i+1,k]}, (-1)^{k-i+1+b} e_{k-i+1,b}] \cdot q\Delta_{[1,k-i],[m-k+i,m]\backslash b} = 0. \tag{28}$$

Proof. Using the Laplace expansion of $q\Delta_{[1,i],[k-i+1,k]}$ along the $k-i+1^{th}$ column, the left hand side of (28) is equal to

$$\begin{aligned}
&\sum_{b\in[m-k+i,m]} [q\Delta_{[1,i],[k-i+1,k]}, (-1)^{k-i+1+b} e_{k-i+1,b}] \cdot q\Delta_{[1,k-i],[m-k+i,m]\backslash b} \\
=&\sum_{b\in[m-k+i,m]} \left[\sum_{c\in[1,i]} (-1)^{k-i+1+c} e_{c,k-i+1} \cdot q\Delta_{[1,i]\backslash c,[k-i+2,k]}, (-1)^{k-i+1+b} e_{k-i+1,b}\right] \\
&\cdot q\Delta_{[1,k-i],[m-k+i,m]\backslash b} \\
=&\sum_{b\in[m-k+i,m]} \sum_{c\in[1,i]} (-1)^{k-i+1+c} [e_{c,k-i+1} \cdot q\Delta_{[1,i]\backslash c,[k-i+2,k]}, e_{k-i+1,b}] (-1)^{k-i+1+b} q\Delta_{[1,k-i],[m-k+i,m]\backslash b}
\end{aligned}$$

By (27) and the fact that $q\Delta_{[1,i]\backslash c,[k-i+2,k]}$ and $e_{k-i+1,b}$ commute, the above expression simplifies to

$$\begin{aligned}
&\sum_{b\in[m-k+i,m]} \sum_{c\in[1,i]} (-1)^{k-i+1+c} [e_{c,k-i+1}, e_{k-i+1,b}] \cdot q\Delta_{[1,i]\backslash c,[k-i+2,k]} (-1)^{k-i+1+b} q\Delta_{[1,k-i],[m-k+i,m]\backslash b} \\
=&\sum_{b\in[m-k+i,m]} \sum_{c\in[1,i]} (-1)^{k-i+1+c} e_{c,b} \cdot q\Delta_{[1,i]\backslash c,[k-i+2,k]} (-1)^{k-i+1+b} q\Delta_{[1,k-i],[m-k+i,m]\backslash b} \\
=&(-1)^{k-i+1} \sum_{b\in[m-k+i,m]} \sum_{c\in[1,i]} (-1)^{k-i+1+c} e_{c,b} \cdot q\Delta_{[1,i]\backslash c,[k-i+2,k]} (-1)^{b} q\Delta_{[1,k-i],[m-k+i,m]\backslash b} \\
=&(-1)^{k-i+1} \sum_{c\in[1,i]} (-1)^{k-i+1+c} q\Delta_{[1,i]\backslash c,[k-i+2,k]} \left(\sum_{b\in[m-k+i,m]} e_{c,b} (-1)^{b} q\Delta_{[1,k-i],[m-k+i,m]\backslash b}\right)
\end{aligned}$$

We now prove that for every value of c, we have

$$\sum_{b\in[m-k+i,m]} e_{c,b} (-1)^{b} q\Delta_{[1,k-i],[m-k+i,m]\backslash b} = 0. \tag{29}$$

We will let B denote the submatrix of E formed by the first $k-i$ rows and the columns $m-k+i, m-k+i+1, \ldots, m$. Then $q\Delta_{[1,k-i],[m-k+i,m]}$ is the determinant of B. As $i \leq k-i$, we see that $e_{c,b}$ is an entry in B. Multiplying the expression in (29) by $(-1)^c$ gives us

$$\sum_{b\in[m-k+i,m]} e_{c,b} (-1)^{c+b} q\Delta_{[1,k-i],[m-k+i,m]\backslash b}. \tag{30}$$

Notice that this is the determinant obtained by the Laplace expansion of the matrix

$$\begin{pmatrix} e_{c,m-k+i} & e_{c,m-k+i+1} & \cdots & e_{c,m} \\ & B & & \end{pmatrix}$$

along its first row. As

$$\begin{pmatrix} e_{c,m-k+i} & e_{c,m-k+i+1} & \cdots & e_{c,m} \end{pmatrix}$$

is the c^{th} row of B, it follows that the expression in (30) is indeed 0. ∎

In the general case, we let $s = |[k-i+1,k] \cap [1,j]|$ and $\{a_1,\ldots,a_s\} = [k-i+1,k] \cap [1,j]$. Following the same method as our previous case, we use the Laplace expansion of $q\Delta_{[1,j],[m-j+1,m]}$ along row a_1, which gives us

$$\begin{aligned}
&[q\Delta_{[1,i],[k-i+1,k]}, q\Delta_{[1,j],[m-j+1,m]}] \\
&= \sum_{b_1\in[m-j+1,m]} [q\Delta_{[1,i],[k-i+1,k]}, (-1)^{a_1+b_1} e_{a_1,b_1} \cdot q\Delta_{[1,j]\backslash a_1,[m-j+1,m]\backslash b_1}] \\
&= \sum_{b_1\in[m-j+1,m]} [q\Delta_{[1,i],[k-i+1,k]}, (-1)^{a_1+b_1} e_{a_1,b_1}] \cdot q\Delta_{[1,j]\backslash a_1,[m-j+1,m]\backslash b_1} \\
&\quad + \sum_{b_1\in[m-j+1,m]} (-1)^{a_1+b_1} e_{a_1,b_1} \cdot [q\Delta_{[1,i],[k-i+1,k]}, q\Delta_{[1,j]\backslash a_1,[m-j+1,m]\backslash b_1}].
\end{aligned} \tag{31}$$

We continue by rewriting the commutator bracket in the second summand of (31) using the Laplace expansion of $q\Delta_{[1,j]\backslash a_1,[m-j+1,m]\backslash b_1}$ along row a_2, which gives us

$$\begin{aligned}
&[q\Delta_{[1,i],[k-i+1,k]}, q\Delta_{[1,j]\backslash a_1,[m-j+1,m]\backslash b_1}] \\
&= \sum_{b_2\in[m-j+1,m]\backslash b_1} [q\Delta_{[1,i],[k-i+1,k]}, (-1)^{a_2+b_2} e_{a_2,b_2} \cdot q\Delta_{[1,j]\backslash\{a_1,a_2\},[m-j+1,m]\backslash\{b_1,b_2\}}] \\
&= \sum_{b_2\in[m-j+1,m]\backslash b_1} [q\Delta_{[1,i],[k-i+1,k]}, (-1)^{a_2+b_2} e_{a_2,b_2}] \cdot q\Delta_{[1,j]\backslash\{a_1,a_2\},[m-j+1,m]\backslash\{b_1,b_2\}} \\
&\quad + \sum_{b_2\in[m-j+1,m]\backslash b_1} (-1)^{a_2+b_2} e_{a_2,b_2} \cdot [q\Delta_{[1,i],[k-i+1,k]}, q\Delta_{[1,j]\backslash\{a_1,a_2\},[m-j+1,m]\backslash\{b_1,b_2\}}].
\end{aligned} \tag{32}$$

We continue this by similarly rewriting $[q\Delta_{[1,i],[k-i+1,k]}, q\Delta_{[1,j]\backslash\{a_1,a_2\},[m-j+1,m]\backslash\{b_1,b_2\}}]$. Reiterating the process, we will eventually get the following commutator bracket inside a nested sum:

$$\begin{aligned}
&[q\Delta_{[1,i],[k-i+1,k]}, q\Delta_{[1,j]\backslash\{a_1,\ldots,a_{s-1}\},[m-j+1,m]\backslash\{b_1,\ldots,b_{s-1}\}}] \\
&= \sum_{b_s\in[m-j+1,m]\backslash\{b_1,\ldots,b_{s-1}\}} [q\Delta_{[1,i],[k-i+1,k]}, (-1)^{a_s+b_s} e_{a_s,b_s} \cdot q\Delta_{[1,j]\backslash\{a_1,\ldots,a_s\},[m-j+1,m]\backslash\{b_1,\ldots,b_s\}}] \\
&= \sum_{b_s\in[m-j+1,m]\backslash\{b_1,\ldots,b_{s-1}\}} [q\Delta_{[1,i],[k-i+1,k]}, (-1)^{a_s+b_s} e_{a_s,b_s}] \cdot q\Delta_{[1,j]\backslash\{a_1,\ldots,a_s\},[m-j+1,m]\backslash\{b_1,\ldots,b_s\}} \\
&\quad + \sum_{b_s\in[m-j+1,m]\backslash\{b_1,\ldots,b_{s-1}\}} (-1)^{a_s+b_s} e_{a_s,b_s} \cdot [q\Delta_{[1,i],[k-i+1,k]}, q\Delta_{[1,j]\backslash\{a_1,\ldots,a_s\},[m-j+1,m]\backslash\{b_1,\ldots,b_s\}}].
\end{aligned} \tag{33}$$

As $[k-i+1,k] \cap [1,j]\backslash\{a_1,\ldots,a_s\} = \emptyset$, it follows that the second summand in (33) equals 0. Thus, to prove that

$$[q\Delta_{[1,i],[k-i+1,k]}, q\Delta_{[1,j],[m-j+1,m]}] = 0,$$

it remains for us to show that the first summand in the expressions in (31), (32), (33), and all other expressions of the same form, equal 0. More explicitly, we need to show that

$$\sum_{b_t\in[m-j+1,m]\backslash\{b_1,\ldots,b_{t-1}\}} [q\Delta_{[1,i],[k-i+1,k]}, (-1)^{a_t+b_t} e_{a_t,b_t}] \cdot q\Delta_{[1,j]\backslash\{a_1,\ldots,a_t\},[m-j+1,m]\backslash\{b_1,\ldots,b_t\}} = 0.$$

This is similar to our proof in Lemma 5.2.1: for conciseness, we let Y_t denote $q\Delta_{[1,j]\backslash\{a_1,\ldots,a_t\},[m-j+1,m]\backslash\{b_1,\ldots,b_t\}}$. Using the Laplace expansion of $q\Delta_{[1,i],[k-i+1,k]}$ along column a_t, we have that

$$\begin{aligned}
&\sum_{b_t\in[m-j+1,m]\backslash\{b_1,\ldots,b_{t-1}\}} [q\Delta_{[1,i],[k-i+1,k]}, (-1)^{a_t+b_t} e_{a_t,b_t}] \cdot Y_t \\
&= \sum_{b_t\in[m-j+1,m]\backslash\{b_1,\ldots,b_{t-1}\}} \sum_{c\in[1,i]} (-1)^{c+a_t} [e_{c,a_t} \cdot q\Delta_{[1,i]\backslash c,[k-i+1,k]\backslash a_t}, e_{a_t,b_t}] \cdot (-1)^{a_t+b_t} \cdot Y_t.
\end{aligned}$$

As $q\Delta_{[1,i]\setminus c,[k-i+1,k]\setminus a_t}$ and e_{a_t,b_t} commute, the above expression simplifies to

$$\sum_{b_t\in[m-j+1,m]\setminus\{b_1,\dots,b_{t-1}\}}\sum_{c\in[1,i]}(-1)^{c+a_t}\cdot e_{c,b_t}\cdot q\Delta_{[1,i]\setminus c,[k-i+1,k]\setminus a_t}\cdot(-1)^{a_t+b_t}\cdot Y_t$$
$$=(-1)^{a_t}\sum_{c\in[1,i]}(-1)^{c+a_t}q\Delta_{[1,i]\setminus c,[k-i+1,k]\setminus a_t}\left(\sum_{b_t\in[m-j+1,m]\setminus\{b_1,\dots,b_{t-1}\}}(-1)^{b_t}e_{c,b_t}\cdot Y_t\right).$$

We now show that for every value of c, we have that

$$\sum_{b_t\in[m-j+1,m]\setminus\{b_1,\dots,b_{t-1}\}}(-1)^{b_t}e_{c,b_t}\cdot q\Delta_{[1,j]\setminus\{a_1,\dots,a_t\},[m-j+1,m]\setminus\{b_1,\dots,b_t\}}=0. \tag{34}$$

Let Y denote the matrix that $q\Delta_{[1,j]\setminus\{a_1,\dots,a_t\},[m-j+1,m]\setminus\{b_1,\dots,b_{t-1}\}}$ is the determinant of. Notice that as $a_1,\dots,a_t\in[k-i+1,k]$, we have $a_1,\dots,a_t>i$. Thus, for every c, e_{c,b_t} is an entry in Y. Multiplying the LHS expression in (34) gives us

$$\sum_{b_t\in[m-j+1,m]\setminus\{b_1,\dots,b_{t-1}\}}(-1)^{c+b_t}e_{c,b_t}\cdot q\Delta_{[1,j]\setminus\{a_1,\dots,a_t\},[m-j+1,m]\setminus\{b_1,\dots,b_t\}}.$$

This is the determinant obtained by the Laplace expansion of the matrix

$$\begin{pmatrix} e_{c,b_{t_1}} & e_{c,b_{t_2}} & \cdots & e_{c,b_{t_{j-t+1}}} \\ & & Y & \end{pmatrix}$$

along its first row. As

$$\begin{pmatrix} e_{c,b_{t_1}} & e_{c,b_{t_2}} & \cdots & e_{c,b_{t_{j-t+1}}} \end{pmatrix}$$

is the c^{th} row of Y, it follows that the expression in (34) is indeed 0. □

6 Further examples of integrable systems for $(\mathcal{R}_n,\{\cdot,\cdot\}_0)$

Kogan and Zelevinsky [2] showed a method for constructing multiple sets of log-canonical functions for $(\mathcal{R}_n,\{\cdot,\cdot\})$. Each construction is associated with a so-called *double reduced word* for (w_0,w_0), where w_0 is the longest element for the Weyl group of GL_n. Rather than recalling all the detailed definitions in [2], we prefer to list the lowest degree terms for some sets of log-canonical functions for $(\mathcal{R}_3,\{\cdot,\cdot\})$ and $(\mathcal{R}_4,\{\cdot,\cdot\})$ that we computed.

From our computation, we found that one can indeed construct an integrable system using the lowest degree terms of the minors associated with other possible double reduced words $\mathbf{i}$ of (w_0,w_0), and the set $\mathbf{C}$. Thus, we made the following conjecture:

Conjecture 6.1. *We denote a double reduced word for (w_0,w_0) for GL_n by $\mathbf{i}$. Let $\mathbf{D_i}$ be any maximal algebraically independent subset of $\{(M_j^{\mathbf{i}})^{low}\}$, where $\{M_j^{\mathbf{i}}\}$ is the set of log canonical functions associated to $\mathbf{i}$. Let $\mathbf{F_i}=\mathbf{D_i}\cup\mathbf{C}$. Then $(\mathcal{R}_n,\{\cdot,\cdot\}_0,\mathbf{F_i})$ is a Liouville integrable system.*

6.1 The case of $\mathcal{R}_3$

We hereby list the lowest degree terms of the minors associated with every possible double reduced word $\mathbf{i}$ of (w_0,w_0). Notice that the subword of $\mathbf{i}$ consisting of the letters 1 and 2 is $(1,2,1)$ or $(2,1,2)$, and the subword consisting of the letters $\overline{1}$ and $\overline{2}$ is $(\overline{1},\overline{2},\overline{1})$ or $(\overline{2},\overline{1},\overline{2})$. By symmetry, we will only consider the following two cases:

1. The subword of $\mathbf{i}$ consisting of the letters 1 and 2 is $(1,2,1)$, and the subword consisting of the letters $\overline{1}$ and $\overline{2}$ is $(\overline{1},\overline{2},\overline{1})$.

2. The subword of $\mathbf{i}$ consisting of the letters 1 and 2 is $(2,1,2)$, and the subword consisting of the letters $\overline{1}$ and $\overline{2}$ is $(\overline{1},\overline{2},\overline{1})$.

For the first case, we list below all the possible double reduced words $\mathbf{i}$ and the lowest degree terms of the minors associated with them.

The double reduced word $\mathbf{i}$	The functions in $\mathbf{Di}$
$(1,2,1,\overline{1},\overline{2},\overline{1})$	$x_{12},x_{13},x_{21},x_{31}$
$(1,2,\overline{1},1,\overline{2},\overline{1})$ $(1,\overline{1},2,1,\overline{2},\overline{1})$ $(1,2,\overline{1},\overline{2},1,\overline{1})$ $(1,\overline{1},2,\overline{2},1,\overline{1})$ $(1,\overline{1},\overline{2},2,1,\overline{1})$	$x_{13},x_{23},x_{31},x_{32}$
$(1,2,\overline{1},\overline{2},\overline{1},1)$ $(1,\overline{1},2,\overline{2},\overline{1},1)$ $(1,\overline{1},\overline{2},2,\overline{1},1)$ $(1,\overline{1},\overline{2},\overline{1},2,1)$	$x_{12},x_{13},x_{31},x_{32}$
$(\overline{1},1,2,1,\overline{2},\overline{1})$ $(\overline{1},1,2,\overline{2},2,\overline{1})$ $(\overline{1},1,\overline{2},2,1,\overline{1})$ $(\overline{1},\overline{2},1,2,1,\overline{1})$	$x_{13},x_{21},x_{23},x_{31}$
$(\overline{1},1,2,\overline{2},\overline{1},1)$ $(\overline{1},1,\overline{2},2,\overline{1},1)$ $(\overline{1},\overline{2},1,2,\overline{1},1)$ $(\overline{1},1,\overline{2},\overline{1},2,1)$ $(\overline{1},\overline{2},1,\overline{1},2,1)$ $(\overline{1},\overline{2},\overline{1},1,2,1)$	$x_{12},x_{13},x_{21},x_{31}$

For the second case, for every possible double reduced word $\mathbf{i}$, the functions in $\mathbf{D_i}$ are $x_{13},x_{21},x_{23},x_{31}$.

6.2 The case of $\mathcal{R}_4$

We hereby show two noteworthy double reduced words $\mathbf{i}$ of (w_0,w_0), and the functions in the set $\mathbf{D_i}$ for each $\mathbf{i}$. Note that for $\mathcal{R}_4$, we have $|\mathbf{C}|=2$, so we want $|\mathbf{D_i}|=8$.

Consider $\mathbf{i}=(1,2,1,3,2,\overline{1},\overline{2},\overline{1},\overline{3},\overline{2},\overline{1},1)$. The lowest degree terms of the minors associated with it are $x_{13},x_{14},x_{23},x_{24},\Delta_{[1,2],[3,4]},x_{41},x_{42},x_{43},\Delta_{[3,4],[1,2]}$. Observe that $x_{13},x_{14},x_{23},x_{24},\Delta_{[1,2],[3,4]}$ are not algebraically independent. As $\mathbf{D_i}$ is one maximal algebraically independent subset of $\{(M_j^{\mathbf{i}})^{\text{low}}\}$, we see that $\mathbf{D_i}$ indeed has 8 functions, namely, one can choose $\mathbf{D_i}$ to be $\{x_{13},x_{14},x_{23},x_{24},x_{41},x_{42},x_{43},\Delta_{[3,4],[1,2]}\}$.

Similarly, for $\mathbf{i}=(\overline{1},1,2,1,3,2,1,\overline{2},\overline{1},\overline{3},\overline{2},\overline{1})$, the lowest degree terms of the minors associated with it are $x_{14},x_{24},x_{34},\Delta_{[1,2],[3,4]},x_{31},x_{32},x_{41},x_{42},\Delta_{[3,4],[1,2]}$, and $\mathbf{D_i}=\{x_{14},x_{24},x_{34},\Delta_{[1,2],[3,4]},x_{31},x_{32},x_{41},x_{42}\}$.

For our future work, we would like to generalise the construction of the Liouville integrable system of $(\mathcal{R}_n,\{\cdot,\cdot\}_0)$ given in Theorem 4.4 by proving Conjecture 6.1. We would also like to complete the quantization of our classical integrable system in Theorem 4.4 by verifying that our candidate for the set $q\mathbf{C}$ satisfies that all elements in $q\mathbf{D}\cup q\mathbf{C}$ commute with each other, and the image of $q\mathbf{C}$ under the isomorphism $\mathrm{gr}(U(\mathfrak{p}))\cong\mathrm{Sym}(\mathfrak{p})$ coincide with $\mathbf{C}$.

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