Knots and Tangles

and how we can untangle them

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Maths Soc Presentation!

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What is a mathematical knot?

Definition

A (mathematical) **knot** is a simple closed curve in $\mathbb{R}^3$ (3-D space).

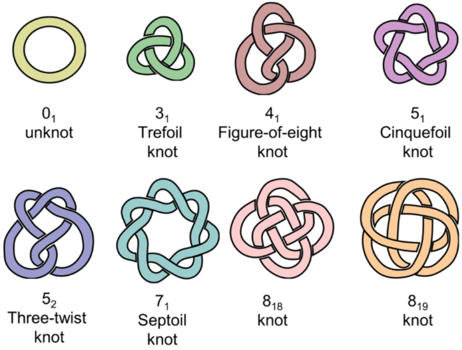

Figure 1: Examples of common knots and their Alexander-Briggs notation

Knots in Human History

Knots have symbolic, religious significance in many cultures:

- Celtic knots: ornamentation of Christian monuments, manuscripts.
- Chinese knots: good-luck charms.

Figure 2: An initial in the Celtic Gospel

Figure 3: Chinese knots

The Quipu

Knots are often used in early number systems; different knots represent different numerical values.

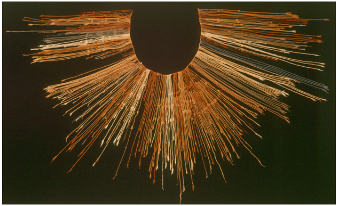

Figure 4: A Quipu from the Andes, 2600 BC

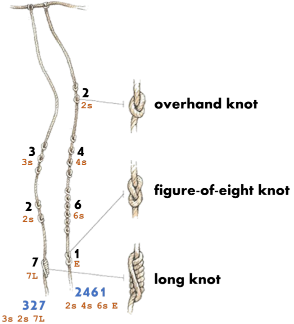

Figure 5: Knots in the Quipu and their numerical values

Knots in Daily Life

- Sailing, rock-climbing
- Knitting
- Organic molecules (e.g., DNA)

Knot Diagrams

Definition

A **diagram of a knot** is the **planar projection** of the knot.
Over-crossings are indicated by solid segments.
Under-crossings are indicated by broken segments.

Figure 6: Figure-of-8 knot in 3D

Figure 7: Diagram of Figure-of-8 knot

When are two knots equivalent?

Definition

Two knots are **equivalent** if one can be deformed smoothly into the other without passing through itself.

Figure 8: Can we 'unknot' this knot?

Reidemeister Moves

Theorem (Reidemeister, 1926)

*Two knots are **equivalent** if and only if they have diagrams that differ by a sequence of the Reidemeister moves.*

Reidemeister Moves

Definition

The **Reidemeister moves** are the following local moves on a knot diagram:

1. R_1: Twist and untwist in either direction.
2. R_2: Move one strand completely over another.
3. R_3: Move a strand completely over or under a crossing.

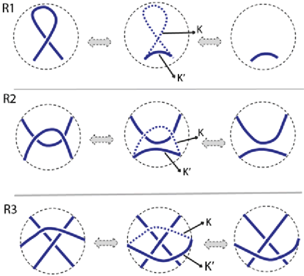

Figure 9: The three Reidemeister moves

Reidemeister Moves

Theorem (Reidemeister, 1926)

*Two knots are **equivalent** if and only if they have diagrams that differ by a sequence of the Reidemeister moves.*

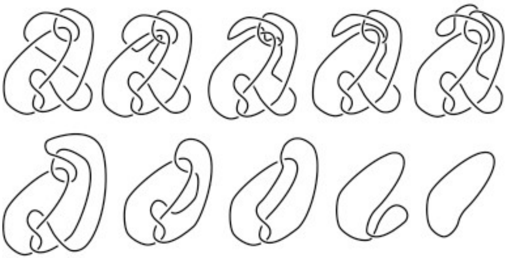

Figure 10

How quickly can we tell if we can 'unknot' the knot?

Theorem (Coward-Lackenby, 2014)

Let D_1 and D_2 be diagrams of some knot in $\mathbb{R}^3$, and let n be the sum of their crossing numbers. Then D_2 may be obtained from D_1 by a sequence of at most $\exp^{(c^n)}(n)$ Reidemeister moves, where $c = 10^{1,000,000}$.

Theorem (Lackenby, 2015)

Let D be a diagram of the unknot with n crossings. Then there is a sequence of at most $(236n)^{11}$ Reidemeister moves that transforms D into the trivial diagram. Moreover, every diagram in the sequence has at most $(7n)^2$ crossings.

P $\neq$ NP?

Definition

An algorithm is said to be of **polynomial time** if its running time is upper bounded by a polynomial expression in the size of the input for the algorithm. That is, $T(n) = O(n^k)$, where k is some nonnegative integer constant and n is the complexity of the input.

Definition

A problem is in the **P (polynomial time)** class if there exists at least one algorithm to solve the problem, such that the number of steps of the algorithm is bounded by a polynomial in n.
A problem is in the **NP (nondeterministic polynomial time)** class if a solution to it can be verified in polynomial time.

P $\neq$ NP?

Corollary

The unknot recognition problem is in NP. That is, we can verify whether a given knot diagram is the diagram of the unknot in polynomial time.

Is unknot recognition in P?

Knot Invariants

- Tricolorability

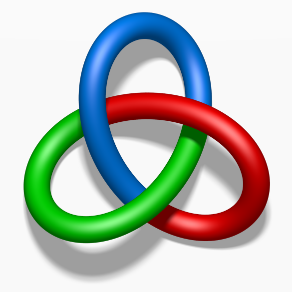

Figure 11: The trefoil knot is tricolourable

- Alexander polynomial

What is a tangle?

Definition

An **n-tangle** is a proper embedding of the disjoint union of n arcs into a 3-ball which sends the endpoints of the arcs to $2n$ marked points on the ball's boundary.

What is a tangle?

We obtain a *n*-tangle by tying *n* strings inside a 3-D ball so that any free ends are on the ball's boundary.

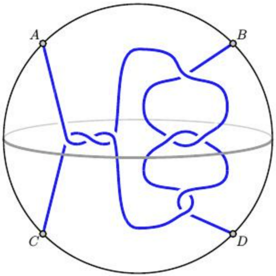

Figure 12: a 2-tangle

A 2-tangle is what we get when we tie 2 ropes together.

Rational tangles

Definition

A **rational tangle** is a 2-tangle that is homeomorphic to the trivial 2-tangle by a map of pairs consisting of the 3-ball and two arcs. It is a 2-tangle formed by a finite sequence of twists and rotations.

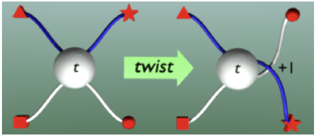

Figure 13: A twist

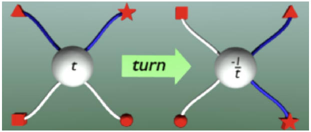

Figure 14: A rotation by 90° clockwise

The Value of a Tangle

Definition (Conway)

The **value** of a 2-tangle is determined by the following algorithm:

- Suppose we start with a tangle of value x.
- If a twist is applied, then $x \to x + 1$.
- If a rotation is applied, then $x \to -\frac{1}{x}$.

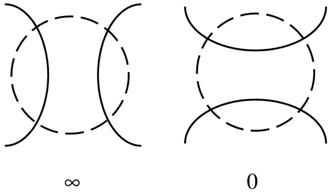

Figure 15: The simplest 2-tangles: the ∞-tangle and the 0-tangle

Let's create a tangle!

- 5 volunteers
- We'll start with the 0-tangle

Untangling Tangles

Figure 16: *Alexander cuts the Gordian Knot*, Donato Creti.

Untangling Tangles

Let's untangle using maths instead:

- Twist and rotation
- We want to end up with the 0-tangle
- Value of tangle $\rightarrow 0$
- 2 cases:
  1. Numerator $>$ denominator
  2. Denominator $>$ numerator

Where did I learn about all of this?

- HCSSiM
- Maths everyday for 6 weeks!
- Application: Essays, problem set

Thank you!