A Mathematical Investigation on Knots and Tangles

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Contents

Conventions

$\mathbb{F}$ denotes either $\mathbb{R}$ or $\mathbb{C}$.
$\mathbb{N}$ denotes the set $\{1, 2, 3, ...\}$ of natural numbers (excluding 0).

1 Knots

1.1 The importance of knots

Knots have been widely used across different cultures throughout human history for both practical and decorative purposes. Quipu, a data recording device that originated from the Andes more than four thousand years ago, utilises knots as a robust measure of data storage in the decimal system.

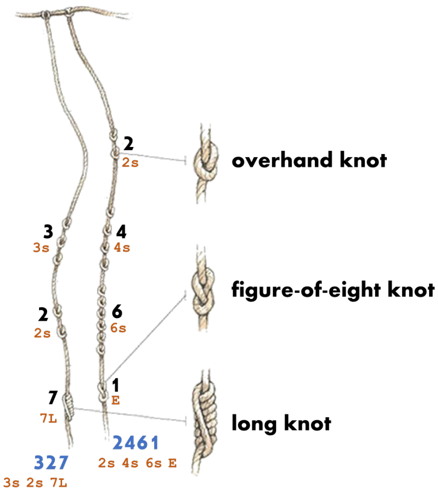

Figure 1: Knots in the Quipu and their numerical value

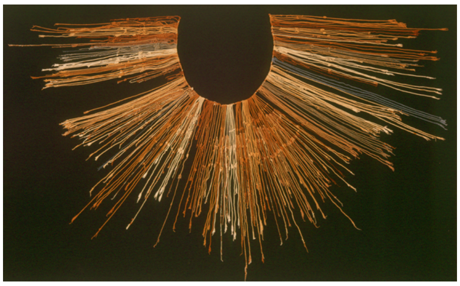

Figure 2: the Quipu

Knots often also have symbolic, religious significance, such as Celtic knots which are extensively used in the ornamentation of Christian monuments and manuscripts, and Chinese knots, which are seen as good luck charms and are used to decorate homes during festivities.

Figure 3:
An initial in the Book of Kells decorated with knots

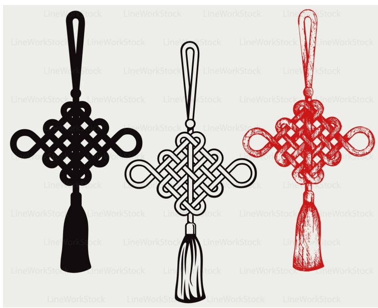

Figure 4: Chinese knots

Figure 5:
Alexander cuts the Gordian Knot, Donato Creti.

An ancient Greek legend associated with Alexander the Great has it that there was a complex knot that tied an oxcart and reputedly, whoever could untie it would be destined to rule all of Asia. Alexander was challenged to untie the knot, but instead of untangling it laboriously as expected, he cut through it with his sword.
Modern-day examples of knots can be found in sailing, rock-climbing, knitting fabrics and organic molecules such as DNA.
The mathematical study of knots began in the 19^{th} century with mathematician Carl Friedrich Gauss. Lord Kelvin's theory that atoms were knots in ether led to Peter Guthrie Tait's creation of the first knot tables for complete classification. Tait, in 1885, published a table of knots with up to ten crossings, and what came to be known as the Tait conjectures. This record motivated the early knot theorists, but knot theory eventually became part of the emerging subject of topology, which studies the properties of a geometric object that are preserved under continuous deformations, such as stretching, twisting, crumpling, and bending; that is, without closing holes, opening holes, tearing, gluing, or passing through itself.

1.2 Some Definitions

Definition 1.1. A (mathematical) **knot** is a simple closed curve in $\mathbb{R}^3$ (3-D space).

Definition 1.2. A knot is **tame** if it can be 'thickened up', that is, there exists an extension to an embedding of the solid torus $S^1 \times D^2$. In this paper, we will only discuss tame knots, so any reference to a knot is a reference to a tame knot.

Examples.

1. The unknot: it is also known as the trivial knot. Intuitively, the unknot is a closed loop of rope without a knot tied into it.
2. The trefoil knot: it is the simplest nontrivial knot. It has three crossings.
3. The figure-eight knot: it is perhaps the most commonly used knot in real life, used in both sailing and rock climbing as a method of stopping ropes from running out of retaining devices.

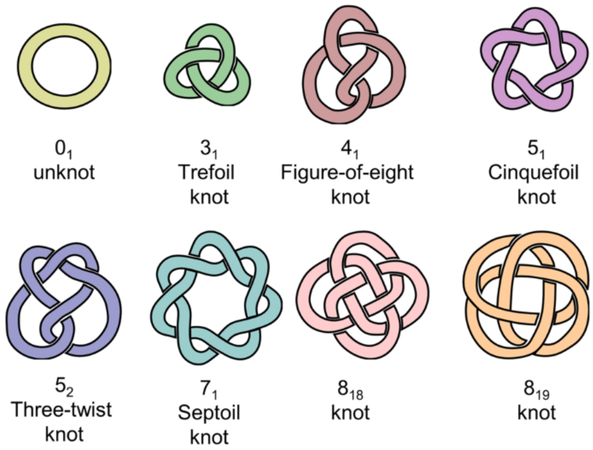

Figure 6: Examples of common knots and their Alexander-Briggs Notation

Non-examples include wild knots, which extend forever, and lengths of rope that do not form closed curves.

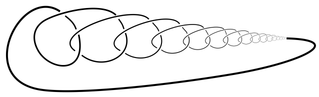

Figure 7: A wild knot

Definition 1.3. Two knots are **equivalent** if one can be deformed smoothly into the other without passing through itself. Topologically speaking, two knots K_1 and K_2 are equivalent if there exists an orientation-preserving PL-homeomorphism $h : S^3 \to S^3$ such that $h(L_1) = L_2$.

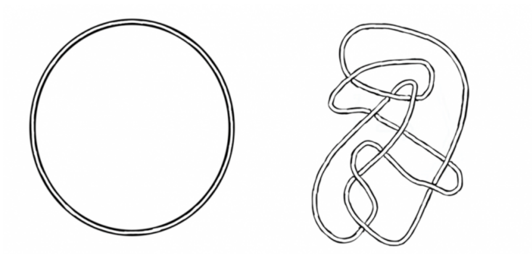

Figure 8: The knot on the right is actually equivalent to the unknot!

Example.

Naturally, we are now wondering whether we can generalize the moves that deform one knot into an equivalent knot, and how we can easily identify whether two knots are equivalent.

1.3 Knot Diagrams

Definition 1.4. Perturb knot K so it only has double points in 2D shadow. The process is called the **planar projection**. The projection with recorded crossings is called a **knot diagram.**

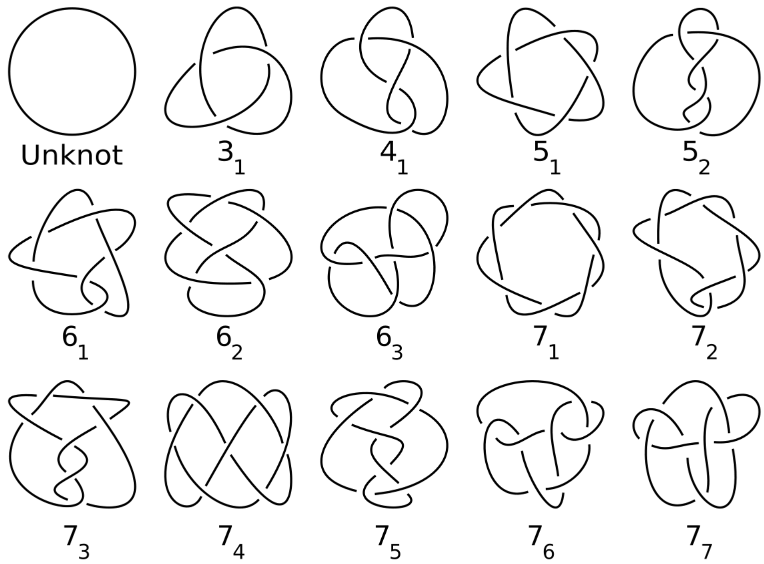

Figure 9: Knot diagrams of common knots

Example. Notice that the diagram indicates which strand is over and which strand is under at a crossing.

Definition 1.5. The **Reidemeister moves** are the following local moves on a knot diagram:

1. Twist and untwist in either direction.
2. The poke move: moving one strand completely over another.
3. Move a strand completely over or under a crossing.

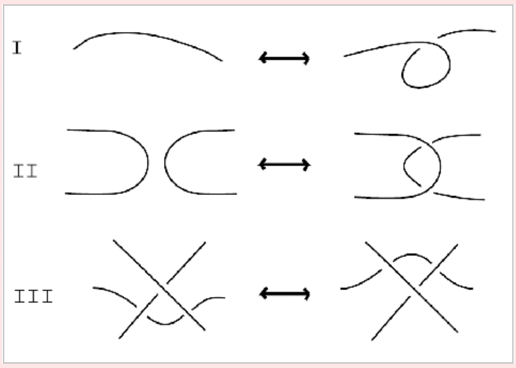

Figure 10: The Reidemeister moves

Theorem 1.6 (Reidemeister, 1926)**.** Knots K and J are equivalent if and only if they have diagrams that differ only by a sequence of the Reidemeister moves.

Remark. Since Reidemeister published his theorem in 1926, it has been speculated as to whether there is an explicit upper bound for the number of moves that are needed, as a function of the number of crossings in the initial and terminal diagrams. In 2014, Coward and Lackenby found an explicit upper bound on how many Reidemeister moves it takes to get between two diagrams of the same knot.

Theorem 1.7 (Coward-Lackenby, 2014)**.** Let D_1 and D_2 be diagrams of some knot in $\mathbb{R}^3$, and let n be the sum of their crossing numbers. Then D_2 may be obtained from D_1 by a sequence of at most $\exp^{(c^n)}(n)$ Reidemeister moves, where $c = 10^{1,000,000}$.

Remark. Subsequently, Lackenby proved a polynomial upper bound on how many Reidemeister moves it takes to reduce a diagram of the unknot to the trivial diagram of an unknot, such as the knot shown in the figure below:

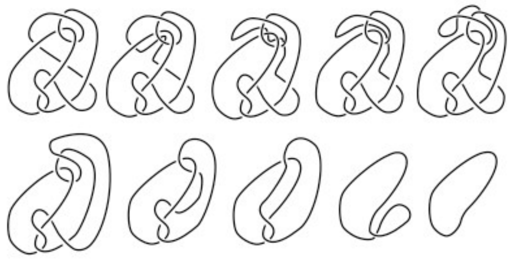

Figure 11: 'Unknotting' a knot that does (k)not look like an unknot at all

Theorem 1.8 (Lackenby, 2015). Let D be a diagram of the unknot with n crossings. Then there is a sequence of at most $(236n)^{11}$ Reidemeister moves that transforms D into the trivial diagram. Moreover, every diagram in the sequence has at most $(7n)^2$ crossings.

Definition 1.9. An algorithm is said to be of **polynomial time** if its running time is upper bounded by a polynomial expression in the size of the input for the algorithm. That is, $T(n) = O(n^k)$, where k is some nonnegative integer constant and n is the complexity of the input.

Definition 1.10. A problem is in the **P (polynomial time)** class if there exists at least one algorithm to solve the problem, such that the number of steps of the algorithm is bounded by a polynomial in n, where n is the length of the input.
A problem is in the **NP (nondeterministic polynomial time)** class if a solution to it can be verified in polynomial time.

Corollary 1.11. The unknot recognition problem is in NP. That is, we can verify whether a given knot diagram is the diagram of the unknot in polynomial time.

Remarks.

1. It remains an unsolved problem whether unknot recognition is in P. Of course, this may be very difficult, because a negative answer would imply that P $\neq$ NP.
2. Similar results on upper bounds on the number of Reidemeister moves needed to obtain an equivalent knot diagram from a given knot diagram have been proven.

2 Tangles

Definition 2.1. An **n-tangle** is a proper embedding of the disjoint union of n arcs into a 3-ball; the embedding must send the endpoints of the arcs to $2n$ marked points on the ball's boundary. In simpler terms, a tangle is a region in a knot or link projection plane surrounded by a circle such that the knot or link crosses the circle exactly four times.

Remark. We can think of a tangle as what we get when we tie two ropes together.
Tangle theory can be considered as an analogue to knot theory, except instead of closed loops, strings whose ends are nailed down are used.

Example. The simplest tangles are the 0-tangle and the ∞-tangle:

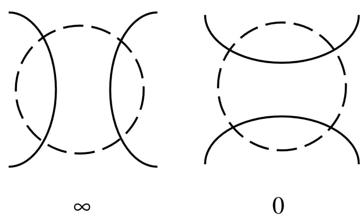

Figure 12: The simplest tangles

2.1 Rational Tangles

Definition 2.2 (Conway). A **rational tangle** is a 2-tangle that is homeomorphic to the trivial 2-tangle by a map of pairs consisting of the 3-ball and two arcs. In other words, it is a 2-tangle that is formed by a finite sequence of 'twists' and 'turns.'

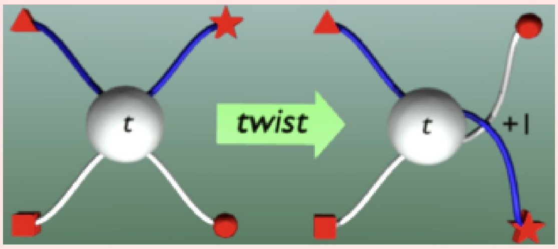

Figure 13: A twist move

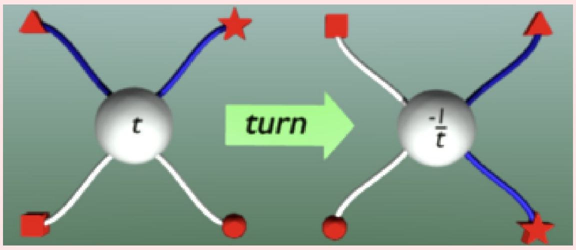

Figure 14: A turn move;for the sake of clarity, we will now call this move 'rotation' instead.

Definition 2.3 (Conway). The **value** of a 2-tangle is determined by the following algorithm:

1. Suppose we start with a tangle of value x, where $x \in \mathbb{Q}$.
2. If a twist is applied, then $x \to x + 1$.
3. If a rotation is applied, then $x \to -\frac{1}{x}$.

Remark. Notice that $\frac{1}{0} = \infty, \frac{1}{\infty} = 0$.

2.2 Interactive Activity

Proposition 2.4. $\forall k \in \mathbb{Q}$, we can create a 2-tangle which has value k.
Conversely, given a rational tangle, we can find its value by recording the moves we used to untangle it.

Proof. We will test this out by creating tangles and untangling them by ourselves! I will be asking for four volunteers to each hold an end of a rope, and record the sequence of moves taken to create our own tangle. This means that we can work out the value of our tangle. Then, we will see how we can see how we can use the given two moves to untangle it by working out how to reduce our value to zero. 😎