The Cayley-Hamilton Theorem
A Topological Proof

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Preliminaries

Let $A \in M_n(K)$.

Definition (Characteristic polynomial for a scalar)

The characteristic polynomial of A for $\lambda \in K$, denoted as $p_A(\lambda)$, is given by $p_A(\lambda) = \det(\lambda I_n - A)$.

If we want to calculate $p_A(M)$ for a matrix M, we can't plug M into the above definition.

Preliminaries

Definition (Characteristic polynomial for a scalar)

The characteristic polynomial of A for $\lambda \in K$, denoted as $p_A(\lambda)$, is given by $p_A(\lambda) = \det(\lambda I_n - A)$.

Expand $\det(\lambda I_n - A)$ and write it as $\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0$.

Definition (Characteristic polynomial for a matrix)

$p_A(M) = M^n + a_{n-1}M^{n-1} + \cdots + a_1 M + a_0 I_n$.

To illustrate...

Example

Consider $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.

$p_A(\lambda) = \det(\lambda I_2 - A) = \det \begin{pmatrix} \lambda - 1 & -2 \\ -3 & \lambda - 4 \end{pmatrix} = \lambda^2 - 5\lambda - 2.$

$p_A(M) = M^2 - 5M - 2I$

To illustrate...

What if we plug A into its own characteristic polynomial?

Example

Consider $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.

$p_A(\lambda) = \det(\lambda I_2 - A) = \det \begin{pmatrix} \lambda - 1 & -2 \\ -3 & \lambda - 4 \end{pmatrix} = \lambda^2 - 5\lambda - 2.$

$p_A(A) = A^2 - 5A - 2 = \begin{pmatrix} 7 & 10 \\ 15 & 22 \end{pmatrix} - \begin{pmatrix} 5 & 10 \\ 15 & 20 \end{pmatrix} - \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

The Theorem

Theorem (Cayley-Hamilton)

For every $A \in M_n(K)$, $p_A(A) = 0$.

Conventional proof

- Linear algebra proofs: (e.g., adjugate matrices proof) computational
- More conceptual proof...as we always want many cool proofs for a cool theorem :D

The Zariski Topology

Definition (Affine space)

The affine space over K is the set of n-tuples with elements in K.
$\mathbb{A}_n = \{(a_1, a_2, ..., a_n) | a_i \in K\}$.

Definition (Algebraic sets)

The algebraic sets of $\mathbb{A}_n$ are of the form
$Z(T) = \{P \in \mathbb{A}_n | f(P) = 0 \ \forall f \in T\}$, where $T \subseteq K[x_1, ..., x_n]$.

Definition (Zariski topology)

We define the Zariski topology on $\mathbb{A}_n$ by taking its closed subsets to be the algebraic sets.

Lemma 1

Lemma (1)

All non-empty open subsets of $\mathbb{A}_n$ are dense.

Definition (Dense)

$A \subseteq X$ is dense in X if $\bar{A} = X$.

Remark

If A intersects every nonempty open subset of X, A must be dense. Otherwise, $U = X \backslash \bar{A} \neq \emptyset$, but as $A \subseteq \bar{A}$, $A \cap U = \emptyset$.

Lemma 1

Proof of Lemma 1.

All open subsets of $\mathbb{A}_n$ are of the form $Z(T)^c$.
Take arbitrary nonempty subsets $Z(T_1)^c, Z(T_2)^c$.
$Z(T_1)^c \cap Z(T_2)^c = (Z(T_1) \cup Z(T_2))^c$.
Points in $Z(T_1) \cup Z(T_2)$ are roots of all $f \in T_1$ or roots of all $g \in T_2$, so they are roots of all $fg \in T_1 T_2$.
$Z(T_1)^c \cap Z(T_2)^c = (Z(T_1 T_2))^c \neq \emptyset$.
$Z(T_1)^c$ intersects every nonempty open subset of $\mathbb{A}_n$, so it is dense. □

Lemma 2

Lemma (2)

Any polynomial map $g : \mathbb{A}_n \to \mathbb{A}_m$ is continuous.

Proof of Lemma 2.

$$
\begin{aligned}
Z(T) &= \{x \in \mathbb{A}_m | f(x) = 0 \ \forall f \in T\}. \\
g^{-1}(Z(T)) &= \{y \in \mathbb{A}_n | g(y) \in Z(T)\} \\
&= \{y \in \mathbb{A}_n | f(g(y)) = 0 \ \forall f \in T\},
\end{aligned}
$$

which is an algebraic, and therefore, closed, subset of $\mathbb{A}_n$. □

Lemma 3

Lemma (3)

For a diagonalisable matrix $M \in M_n(K)$, $p_M(M) = 0$.

Proof of Lemma 3.

$M = P^{-1}BP$, where B is a diagonal matrix.

$$\begin{aligned} p_M(\lambda) &= \det(\lambda I - M) = \det(P^{-1}\lambda I P - P^{-1}BP) \\ &= \det(P^{-1}(\lambda I - B)P) = \det(\lambda I - B) = p_B(\lambda) \end{aligned}$$

Let the diagonal entries of B be $\lambda_1, ..., \lambda_n$. They are also the eigenvalues of B.
$p_B(\lambda) = \prod_{i=1}^{n}(\lambda - \lambda_i)$, so
$p_B(B) = \prod_{i=1}^{n}(B - \lambda_i I) = 0 = p_M(M)$. □

What we've shown so far

Lemma (1)

All non-empty open subsets of $\mathbb{A}_n$ are dense.

Lemma (2)

Any polynomial map $g : \mathbb{A}_n \to \mathbb{A}_m$ is continuous.

Lemma (3)

For a diagonalisable matrix $M \in M_n(K)$, $p_M(M) = 0$.

Let D be the set of diagonalisable matrices in $M_n(K)$.
We can think of $M_n(K)$ as $\mathbb{A}_{n^2}$, so $D \subset \mathbb{A}_{n^2}$.

Proof of Theorem

Proof.

D^c is the set of matrices with repeated eigenvalues.
Their characteristic polynomials have repeated roots, so their discriminant ($\in K[x_1, ..., x_{n^2}]$) is zero. I.e.,
$D^c = \{M \in M_n(K) | \Delta(p_M) = 0\}$, which is closed in $\mathbb{A}_{n^2}$.
Hence, D is dense.
Define $\phi : \mathbb{A}_{n^2} \to \mathbb{A}_{n^2}$ by $\phi(A) = p_A(A)$.
ϕ is continuous and vanishes on the dense subset D, so ϕ vanishes everywhere in $\mathbb{A}_{n^2}$. □

Further links

The Zariski dense argument is very powerful!

Example

$$\det(I - AB) \stackrel{?}{=} \det(I - BA)$$

Equality clearly holds when A, B are invertible...

- The set of invertible matrices (i.e., matrices with nonzero determinants), is also dense.
- To verify a polynomial identity on $M_n(K)$, it suffices to check the case for the diagonalisable or the invertible matrices, which is much easier!

References

1. Hartshorne, R. (2010) *Algebraic geometry*. New York: Springer.
2. `https://cykenleung.blogspot.com/2012/12/a-proof-of-cayley-hamilton-theorem.html` (An outline of the last proof)

Thank you!